Spectral Concentration for High Contrast Random Media

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1 Introduction

(a) We investigate spectral properties of random selfadjoint divergence type operators

$$T_\omega = \nabla^* a_\omega(x)\nabla = -\sum_k \partial_k\big(a_\omega(x)\partial_k\big) \qquad \text{in } L_2(\mathbb{R}^m) \tag{1.1}$$

with $m \geq 2$. The scalar valued coefficient functions a_ω are assumed to be ergodic with respect to shifts given by translations in $\mathbb{R}^m$ (in some cases only $\mathbb{Z}^m$-lattice translations are allowed). Well known results (see, e.g., [Pa, PF, KM]) imply that both the spectrum and the density of states of T_ω are almost surely independent of ω, i.e. they are non-random. We intend to determine some properties of these almost sure characteristics.

Throughout the present work we consider a special kind of coefficients a_ω, which is mainly motivated by work of Hempel and Lienau [HL1, HL2]. These authors investigate *periodic* divergence type operators

$$T_\lambda^{\mathrm{per}} = \nabla^* a_\lambda^{\mathrm{per}}(x)\nabla \qquad \text{in } L_2(\mathbb{R}^m),\ \lambda \geq 1, \tag{1.2}$$

similar to operators in (1.1), but with a_λ^{per} non-random given by

$$a_\lambda^{\mathrm{per}}(x) = \begin{cases} 1 & \text{if } x \in M, \\ \lambda & \text{otherwise,} \end{cases} \qquad \lambda \geq 1, \tag{1.3}$$

where $M \subset \mathbb{R}^m$ is a periodic set with respect to the $\mathbb{Z}^m$-lattice, say. Moreover, $\mathbb{R}^m \setminus M$ is assumed to be connected (complete assumptions are listed in Chapter 4). The principal attention lies in situations for very large λ and in the limit case $\lambda \to \infty$. This kind of operators can be interpreted as models of high contrast periodic media. Spectral properties then classify some properties of the medium, like in the field of heat conduction or acoustics (cf. [FKu2]). There are also applications in photonic crystals (cf. [JMW, FKu2]), in some special cases.

This work can be viewed as a combination of random operators on the one hand, and perturbations of operators in the large coupling limit on the other. The latter has been studied in many situations. For the Schrödinger case we refer, e.g., to [HZh, AB], some related results can also be found in [HH1]. In a similar (but non-random) setting as in the present work, namely for operators $-\Delta + \lambda\chi_{\mathbb{R}^m\setminus M}$, $M \subset \mathbb{R}^m$, one gets a limit operator in the strong resolvent sense,

$$-\Delta + \lambda\chi_{\mathbb{R}^m\setminus M} \longrightarrow -\Delta_M \quad \text{in s.r.s.}, \quad \lambda \to \infty. \tag{1.4}$$

Here $-\Delta_M$ denotes the Dirichlet Laplacian on the set M. An analogous result holds for magnetic Hamiltonians (under additional assumptions),

$$(-i\nabla - \lambda\vec{a})^2 \longrightarrow -\Delta_M \quad \text{in s.r.s.}, \quad \lambda \to \infty, \tag{1.5}$$

where $M = \{x \in \mathbb{R}^m \mid d\vec{a} = 0\}$. For details see [HH1, HH2], which contains also further results on norm resolvent convergence and concentration regions of the spectrum.

An interesting characteristic is the existence or non-existence of gaps in the spectrum. Values inside a spectral gap correspond to forbidden energies, or frequencies, or modes. A spectral gap occurs if and only if the density of states vanishes in that specific region. Here an inconsistency arises between the mathematical point of view on the one hand, and the physical point of view on the other hand: an experimentalist just can detect spectrum in a given region if the density of states has a sufficiently high amount there. Therefore we have to look also at regions where the density of states is very small, but still positive. In this situation we speak of *pseudo-gaps*, although there is spectrum in that region.

The work of Hempel and Lienau [HL1, HL2] is mainly directed at the existence of gaps and pseudo-gaps together with their location, as the contrast parameter λ gets very large. For a brief review of their results see Chapter 4.

From a geometric point of view, operators given by (1.2) and (1.3) describe the following situation, thinking in terms of heat conduction: a strongly conductive base medium, namely the region $\mathbb{R}^m \setminus M$ where we have the value λ, is perturbed by areas of some fixed lower conductivity, namely M where the coefficients take the fixed value 1. We can, for instance, interpret $\mathbb{R}^m \setminus M$ as a connected metal, and M as a union of impurities, like grains of sand or air bubbles. This suggests to assume a special structure of the perturbing set M. Grains should be connected and do not enclose some metal, i.e. the metal region $\mathbb{R}^m \setminus M$ is connected too. Moreover, different grains should not touch each other, i.e. at least a thin film of the base medium separates them. Summarizing, the impurities are given by a countable set of grains,

$$M = \bigcup_{i \in \mathbb{J}} M_i, \qquad \inf_{i \neq j} \operatorname{dist}(M_i, M_j) > 0. \tag{1.6}$$

Of course, in the $\mathbb{Z}^m$-periodic case the individual grains M_i are given by periodically repeated copies of a suitable primary grain Ξ,

$$M_i = \Xi + i \qquad \text{for } i \in \mathbb{Z}^m. \tag{1.7}$$

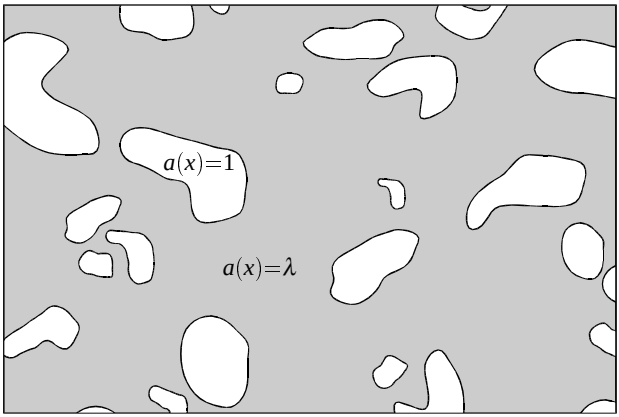


Figure 1.1: Randomly distributed grains of variable size and shape.

While periodicity is a very strong assumption, it would lead to more realistic models if one allows for random models. And this is exactly what we will do in the present work.

With respect to the existence of limit operators as in Eqns. (1.4) and (1.5), random settings differ from non-random ones in an important manner: while both the spectrum and the density of states are almost surely independent of ω, we cannot expect a unique (non-random) limit operator to exist, because each realization T_ω, i.e. ω fixed, provides its own limit operator. Although these limit operators differ, they have one thing in common, namely their spectral properties are the same, almost surely. Therefore, we focus our attention on the almost sure spectrum and the almost sure density of states.

(b) To describe our main results we first give a brief idea of our random models. Note that we have reduced this to a purely geometric problem, namely the construction of mutually non-intersecting random grains, forming the set M. For a detailed description see Chapter 3. We aim to realize models suggested by Fig. 1.1, showing "randomly distributed grains" in an intuitive manner. While our models are neither periodic nor locally uniform, they have one property in common, that is they are macroscopically uniform or self similar (cf. [EK]), or, in other words, ergodic. This is exactly what one naturally would expect from well mixed materials, like random alloys or foamed materials.

In **model M1** we restrict to a spatial generalization of periodicity. That is, in Eqn. (1.7) replace the translation vectors $i \in \mathbb{Z}^m$ by an appropriate (random)

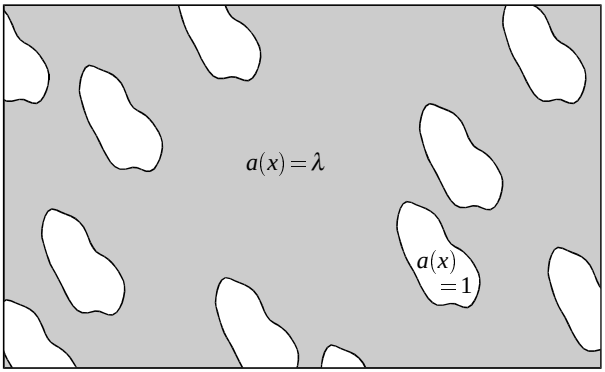


Figure 1.2: Randomly distributed grains of a fixed type.

point process $x_i(\omega)$ and leave Ξ invariant,

$$M_i(\omega) = \Xi + x_i(\omega), \qquad i \in \mathbb{N},$$

yielding grain models as in Fig. 1.2.

In the opposite direction, periodic models can be generalized as follows. In analogy to the periodic case, choose a random primary grain $\Xi(\omega)$ (whatever that means) instead of a fixed on, and replace Ξ by independent copies of $\Xi(\omega)$ in Eqn. (1.7),

$$M_i(\omega) = \Xi_i(\omega) + i, \qquad i \in \mathbb{Z}^m.$$

This is roughly what we call **model M2**, Fig. 3.3 on page 24 shows a sample. One can think of model M2 as a periodic structure which is slightly perturbed.

Combining both ideas described above, we obtain **model M3** which performs essentially what we had in mind (cf. Fig. 1.1). The key point in constructing model M3 are so called soft-core models that occur in stochastic geometry. We describe these models in Section 3.3 in more detail. Mainly, soft-core processes model random arrangements of mutually non-intersecting balls with variable (meaning random) radii (see Fig. 3.5 on page 27). This provides the starting point in the construction of mutually non-intersecting grains.

Our main results are as follows.

Theorem A (see Theorems 5.1, 5.2 and 5.6)

(a) In case of models M1 and M3 the almost sure spectrum Σ_λ does not depend on λ, in particular, we have

$$\Sigma_\lambda = [0,\infty), \qquad \textit{all } \lambda \geq 1.$$

(b) The almost sure spectrum Σ_λ of model M2 has a band–gap structure. Moreover, under some additional assumptions on the grain distribution, spectral gaps open up in Σ_λ, for λ large.

Particularly part (a) of Theorem A makes it clear why it is important to investigate the density of states. Looking at pseudo-gaps and concentration regions of the density of states, it only makes sense to do so for the limit procedure $\lambda \to \infty$. It turns out that in all models the density of states measure μ_λ converges (vaguely in the sense of measures) as $\lambda \to \infty$, and that μ_λ concentrates its mass near the Dirichlet eigenvalues of the (perhaps random) grains. The reason for this lies in a rather general convergence of operators (cf. [HL1] and Section 3.4). Suppose a deterministic, hence non-random, setting: let M denote a union of (non-random) grains as in Eqn. (1.6) and T_λ the corresponding operator (not necessarily periodic) according to Eqns. (1.2) and (1.3). Then T_λ converges, as $\lambda \to \infty$, in strong resolvent sense to $-\Delta_M$, the Laplacian in $L_2(M)$ with Dirichlet boundary conditions,

$$T_\lambda \longrightarrow -\Delta_M \quad \text{in s.r.s.}, \qquad \lambda \to \infty. \tag{1.8}$$

Since the individual grains M_i do not intersect, we get

$$-\Delta_M = \bigoplus_{i\in\mathcal{I}} -\Delta_{M_i}, \quad \text{and} \quad \sigma(-\Delta_M) = \bigcup_{i\in\mathcal{I}} \sigma(-\Delta_{M_i}), \tag{1.9}$$

which is a discrete set. Therefore, together with Theorem A, this implies that T_λ does not converge in norm resolvent sense, because $\sigma(T_\lambda)$ does not converge to $\sigma(-\Delta_M)$. But convergence in strong resolvent sense is strong enough to establish convergence of the density of states.

Theorem B (see Theorem 5.9) *For the almost sure density of states measure μ_λ in models M1–M3 we have*

$$\mu_\lambda \longrightarrow \mathbb{E}\big[\mu(-\Delta_{M(\omega)})\big] \quad \textit{vaguely}, \qquad \lambda \to \infty, \tag{1.10}$$

where $\mu(-\Delta_{M(\omega)})$ denotes the density of states measure of the Dirichlet Laplacian $-\Delta_{M(\omega)}$ on $M(\omega)$.

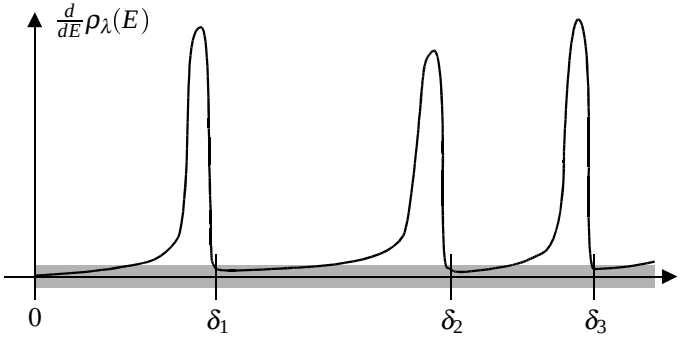


Figure 1.3: Density function of the integrated density of states ρ_λ of model M1.

Note that Eqn. (1.9) and the RHS of (1.10) only depend on the shape of the grains and not on their location. Thus, for example, as in model M1 the grain shapes are non-random and $\sigma(-\Delta_{M(\omega)}) = \sigma(-\Delta_\Xi)$ is purely discrete, the almost sure density of states behaves, for large λ, as is pictured in Fig. 1.3. The graph is to be understood as an attempt to illustrate the density $\frac{d}{dE}\rho_\lambda(E)$ of the integrated density of states $\rho_\lambda(E) = \mu_\lambda\big((-\infty, E)\big)$. The shaded area suggests the almost sure spectrum $\Sigma_\lambda = [0, \infty)$. Here we denote $\sigma(-\Delta_\Xi) = \{\delta_k \mid k \in \mathbb{N}\}$, the Dirichlet eigenvalues of the Laplacian on the (non-random) primary grain Ξ. While the concentration points coincide with those of the periodic case, the spectrum exhibits no gaps at all, in contrast to the periodic case (cf. Fig. 1.3 and Fig. 4.2 on page 35).

In the case of model M2 we expect, generally, concentration intervals, instead of discrete points. Fig. 1.4 shows a typical situation for large λ, the spectral bands are denoted by $[\alpha_{k,\lambda}, \beta_{k,\lambda}]$.

We can regard model M3 as a combination of both previous models. Interestingly enough, the density of states concentrates in the same way as in model M2, while the spectrum behaves as in model M1. For further details see Theorem 5.9.

To summarize, for large λ, in model M1 there always exist pseudo-gaps, while in models M2 and M3 this holds only under additional assumptions on the grain-shape distribution. Moreover, only in model M2 there is a chance to have real gaps in the spectrum (see Corollary 5.5).

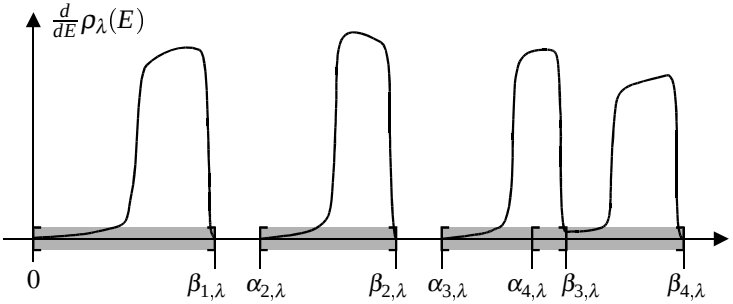


Figure 1.4: Density function of the integrated density of states ρ_λ of model M2.

More generally, one could also study operators with coefficient matrices

$$\mathbf{a}_\lambda(x) = \begin{cases} \mathbf{b}(x) & \text{if } x \in M, \\ \lambda \mathbf{c}(x) & \text{otherwise,} \end{cases} \qquad \lambda \geq 1,$$

replacing Eqn. (1.3), where $\mathbf{b}, \mathbf{c}$ are appropriate positive definite matrices. But there will not be an essential change of results; in fact, besides some additional technical overhead, the limits corresponding to Eqns. (1.8)–(1.10) will then be determined via the Dirichlet operator $\nabla^* \mathbf{b}(x) \nabla$ in $L_2(M)$.

On first sight, the results presented here seem natural extensions of the previously known behaviour of homogeneous or periodic media. But the introduction of randomness, together with stochastic modelling of appropriate disordered media, is to be understood as a mathematical challenge.

(c) The present work is organized as follows. First, in Chapter 2, we introduce random divergence type operators together with some basic facts. In Chapter 3 we give a detailed description of our random grain models, mentioned in part (b) above. This is strongly connected with point processes and germ-grain models such as hard-core and soft-core processes known form stochastic geometry ([SKM, StSt, St1, St2, St3]). Moreover, we state some results that help to determine the almost sure spectrum by periodic approximation. Thus, to understand the random case of grain impurities one first has to look at the (non-random) periodic case. In our special situation this is due to Hempel and Lienau [HL1], and we give a brief review in Chapter 4. The last chapter then contains the proofs of our main results stated above.

(d) In the context of divergence type operators there exists some literature that is connected with the present work. Our interest is mainly moti-

vated by work of Alama et al. [AADH] and Hempel [H]. Figotin and Kuchment [FKu1, FKu2, FKu3] treat also the periodic case with high contrast, but they consider a different limit procedure and uses tensor products of one dimensional systems to generate band-gaps. Figotin and Klein [FKl2] deals also with periodic divergence type and Maxwell operators (photonic crystals), they investigate the arising of defect modes caused by local impurities. There is also a connection to periodic Laplace-Beltrami operators ([DH, G]).

Some results on random divergence type operators can be found in Figotin and Klein [FKl1], here the main interest is localization. These authors consider a random situation which is similar to random alloy type potentials in the context of Schrödinger operators, e.g.

$$-\Delta + V_\omega(x), \qquad V_\omega(x) = \sum_{i \in \mathbb{Z}^m} q_i(\omega) f(x-i),$$

where $f(y)$ decays fast enough as $|y| \to \infty$, and q_i denote i.i.d. random variables. Our models differ in the sense that, essentially, we pay more attention to geometric purposes. Random Schrödinger operators have received much more attention, see, e.g., [Pa, KM, Ki2, PF] and references therein.

Hempel and Kirsch [HK] study random distributions of periodic Schrödinger operators. They obtain a concentration result for the density of states when the spatial intensity of impurities tends to zero. This limit, acting only on the probabilistic intensity, contrasts in some sense our limit, that acts only on the growth of the coefficients and not the random models.

A further geometrical approach to random media can be found in the monograph by Sznitman [Sz]. He considers random Schrödinger operators where the potential is given by the sum of randomly distributed copies of a fixed potential well,

$$-\Delta + V_\omega(x), \qquad V_\omega(x) = \sum_{i \in \mathbb{N}} f\big(x - x_i(\omega)\big).$$

These obstacles are distributed according to a Poisson point process $x_i(\omega)$ (cf. Section 3.1), such that potential wells, which lie very close together, might interact, forming a new joined type of potential well. While this overlapping or adding makes sense in case of potentials, we have to be more carefully in constructing our random grain models to avoid overlapping.

Acknowledgements

It is a pleasure for me to thank R. Hempel, my thesis advisor, for his kind constant attention and numerous useful discussions. He took always care, not

only in mathematical but also in personal matter, for which I am very grateful. I would also like to thank G. Last who drew my attention to some topics of stochastic geometry.

2 Ergodic operators

The goal of this chapter is a brief introduction to the theory of ergodic families of selfadjoint operators. First we introduce the usual stochastic machinery to define ergodic operators, then spectral properties like the spectrum and the density of states are discussed. Finally we look at approximations of ergodic operators by certain periodic ones.

2.1 Ergodic families of selfadjoint operators

To define ergodic stochastic processes we first need some notation. In this chapter we follow the review of Kirsch in [Ki2]. We also refer to this work or references therein for proofs. Another brief review can be found in [AGHH]. Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space and X_i ($i \in \mathcal{I}$) a stochastic process. Furthermore, suppose the state space F to be a polish space. Hence we may assume Ω to be the canonical probability space with $\mathcal{F}$ the σ-algebra generated by the cylinder sets. Thus, X_i is just given by

$$X_i(\omega) = \omega(i), \qquad \text{for } i \in \mathcal{I},\ \omega \in \Omega, \tag{2.1}$$

and P coincides with the distribution of the process $X = \{X_i\}$, i.e. $P = P_X$. Since we need a spatial comparison operator, suppose the index set $\mathcal{I}$ to be an abelian group (in all our applications we set $\mathcal{I} = \mathbb{Z}^m$ or $\mathcal{I} = \mathbb{R}^m$). The *shifts* (or *translations*) $\{\tau_i\}_{i\in\mathcal{I}}$ are defined by

$$\tau_i : \Omega \to \Omega, \quad (\tau_i\omega)(j) := \omega(j-i), \qquad i, j \in \mathcal{I},\ \omega \in \Omega. \tag{2.2}$$

Measurable shifts τ_i are called *measure preserving* if $P(\tau_i^{-1}A) = P(A)$ for all $A \in \mathcal{F}$. We say a set $A \in \mathcal{F}$ is *invariant* (under $\{\tau_i\}$) if $\tau_i^{-1}A = A$ for all $i \in \mathcal{I}$. Now we are prepared to define ergodicity.

Definition 2.1

(a) A measure preserving family of shifts $\{\tau_i\}_{i\in\mathcal{I}}$ is called *ergodic* (or an *ergodic family of shifts*) if any invariant $A \in \mathcal{F}$ has probability $P(A)$ equal to 0 or 1.

(b) We call a stochastic process $\{X_i\}_{i\in\mathcal{I}}$ *ergodic* if the corresponding shifts $\{\tau_i\}$ are ergodic.

This definition might suggest ergodicity being a property of the shifts instead of the process. But ergodicity of the shifts is defined as a property of the

probability measure P, and since we choose $P = P_X$ it is in fact a property of the stochastic process $\{X_i\}$. Often one uses a stronger property than ergodicity, namely *mixing*. Consider a process X_i with index set $\mathcal{I} = \mathbb{Z}^m$ or $\mathcal{I} = \mathbb{R}^m$. Then the random field is said to be *mixing* if for all $A, B \in \mathcal{F}$

$$P\big(A \cap \tau_i^{-1} B\big) \longrightarrow P(A)P(B), \qquad \text{as } |i| \to \infty. \tag{2.3}$$

Roughly speaking, this condition says that the correlation of two cylinder sets A and B vanishes provided that the distance between their supporting points (or indices) tends to infinity.

We now consider families $\{T_\omega\}_{\omega\in\Omega}$ of P-almost surely (P-a.s.) selfadjoint operators in a given separable Hilbert space $\mathcal{H}$. Such a family is called *measurable* if the family of its resolvents is weakly measurable, i.e. for all $u, v \in \mathcal{H}$ the scalar product $\omega \mapsto \big((T_\omega + i)^{-1}u, v\big) \in \mathbb{C}$ is measurable.

Definition 2.2 Let $\{T_\omega\}_{\omega\in\Omega}$ denote a family of P-a.s. selfadjoint operators in a separable Hilbert space $\mathcal{H}$. Furthermore let $\{\tau_i\}_{i\in\mathcal{I}}$ be an ergodic family of shifts. Then we call $\{T_\omega\}$ *ergodic* (or an *ergodic family of operators*) if it is measurable and there exist unitary operators U_i in $\mathcal{H}$ such that

$$T_{\tau_i\omega} = U_i\, T_\omega\, U_i^{-1} \tag{2.4}$$

for all $i \in \mathcal{I}$ and $\omega \in \Omega$.

One can think of ergodicity as a generalization of periodicity in the following sense. If U_i are the unitary operators which correspond to translations according to a given lattice Γ, then a Γ-periodic operator T satisfies $T = U_i T U_i^{-1}$. This looks like (2.4), except that periodic operators remain the same when they are shifted while ergodic operators only remain the same up to unitary equivalence. But as spectral theory cannot distinguish between unitarily equivalent operators, it is not too surprising that several spectral properties of ergodic operators do not depend on a specific realization ω almost surely. This will be expressed in more detail in the next sections.

2.2 Ergodic divergence type operators

Throughout this work we consider selfadjoint divergence type operators

$$T = \nabla^* a(x) \nabla = -\sum_{k=1}^{m} \partial_k \big(a(x)\partial_k\big)$$

acting in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m)$, where the function $a(x)$ is measurable, real valued, and uniformly positive definite, $0 < c_1 \le a(x) \le c_2 < \infty$. As usual, these s.a. operators are defined via quadratic forms. In $\mathcal{H}$ we define the quadratic form

$$\mathbf{t}[u] = (a\nabla u, \nabla u) = \int_{\mathbb{R}^m} a(x)\nabla u(x) \cdot \nabla \overline{u}(x) dx \tag{2.5}$$

for $u \in D(\mathbf{t}) = \mathcal{H}^1(\mathbb{R}^m)$, the usual Sobolev space consisting of those functions $u \in \mathcal{H}$ that have a square integrable distributional gradient, equipped with the canonical norm $\|u\|_1 = \left(\|u\|^2 + \|\nabla u\|^2\right)^{1/2}$. The form $\mathbf{t}$ is positive, densely defined and closed, hence it defines a unique selfadjoint operator T in $\mathcal{H}$ by

$$(Tu, v) = \mathbf{t}[u, v], \qquad u \in D(T),\ v \in D(\mathbf{t}).$$

As for random models, we consider random coefficients $a_\omega(x)$ as a stochastic process $X_x(\omega) = a_\omega(x)$ with index set $\mathcal{I} = \mathbb{R}^m$. This bewildering notation results in a change of viewpoint: in functional analysis one would naturally think of random coefficients as functions $a_\omega(\cdot)$ for each ω, while in probability theory the coefficients are considered as functions $a_\cdot(x)$ with parameter x.

Denote the corresponding operators by T_ω, and assume that each a_ω belongs to a proper (polish) function space. Our notation is based on the concept of canonical probability spaces, hence combining Eqns. (2.1) and (2.2) gives

$$a_{\tau_i\omega}(x) = (\tau_i\omega)(x) = \omega(x - i) = a_\omega(x - i). \tag{2.6}$$

The unitary family U_i in $L_2(\mathbb{R}^m)$ is given by translations $\big(U_i u\big)(x) = u(x - i)$, for $i \in \mathcal{I} = \mathbb{R}^m$, implying

$$\big(U_i T_\omega U_i^{-1}\big)u = U_i\big(\nabla^* a_\omega(\cdot)\nabla u(\cdot + i)\big) = \nabla^* U_i\big(a_\omega(\cdot)U_i^{-1}(\nabla u)(\cdot)\big) = T_{\tau_i\omega}u,$$

where we have used Eqn. (2.6). If the shifts τ_i are ergodic all requirements of Definition 2.2 are fulfilled, provided $\omega \mapsto a_\omega$ is measurable (cf. Appendix A.1), so that T_ω is an ergodic family of operators.

For an example, take $a \in L_1(\mathbb{R}^m)$ with compact support and independent identically distributed random variables $q_i(\omega)$, $i \in \mathbb{Z}^m$, with common distribution P_0 such that the support of P_0 is contained in a compact interval K. Moreover, let P be the product measure $\bigotimes_{i\in\mathbb{Z}^m} P_0$ on $\Omega = K^{\mathbb{Z}^m}$. Due to independence, it is easily seen that $\{q_i\}$ is ergodic with index set $\mathcal{I} = \mathbb{Z}^m$. If we set

$$a_\omega(x) = \sum_{i\in\mathbb{Z}^m} q_i(\omega)a(x - i) \in L_{1,\mathrm{loc}}(\mathbb{R}^m) \tag{2.7}$$

we encounter a problem: a_ω fails to be ergodic because Eqn. (2.6) holds only for $i \in \mathbb{Z}^m$ (and not $i \in \mathbb{R}^m$). But we can avoid this difficulty by calling such random coefficients $\mathbb{Z}^m$-ergodic if there are ergodic shifts τ_i, $i \in \mathbb{Z}^m$, with

$$a_{\tau_i\omega}(x) = a_\omega(x-i), \qquad i \in \mathbb{Z}^m,\ x \in \mathbb{R}^m. \tag{2.8}$$

With this notation a_ω from Eqn. (2.7) is $\mathbb{Z}^m$-ergodic and we call the corresponding divergence type operators $T_\omega = \nabla^* a_\omega \nabla$ ergodic with respect to $\mathbb{Z}^m$ (or $\mathbb{Z}^m$-ergodic); measurability of T_ω is treated in [FKl1], Appendix A.

It is well known (cf. [Ki2, Ki1]) that all properties of ergodic operators hold also for $\mathbb{Z}^m$-ergodic operators; that is why we drop this difference and speak just of *ergodic* operators.

Most results stated in the sequel hold not only for random divergence type operators but also for more general operator classes. Nevertheless, as we are only interested in divergence type operators, we are going to deal with nothing but operators of this class for the rest of this work.

2.3 Spectral properties

Now we will look at some spectral properties of ergodic operators. More precisely, we state some well known results about the spectrum and the density of states. We start with the spectrum.

Theorem 2.3 (Pastur) *Let $\{T_\omega\}$ be ergodic. Then the spectrum $\sigma(T_\omega)$ of T_ω is P-almost surely non-random. More precisely, there exists a (non-random) set $\Sigma \subset \mathbb{R}$ such that*

$$P\Big(\sigma(T_\omega) = \Sigma\Big) = 1.$$

A proof can be found in [Pa, AGHH, Ki2]. Analogous results as in Theorem 2.3 hold not only for the entire spectrum but for several parts of the spectrum. It is easy to see that both the essential spectrum $\sigma_{\mathrm{ess}}(T_\omega)$ and the discrete spectrum $\sigma_{\mathrm{dis}}(T_\omega)$ are non-random sets almost surely. The proofs for the continuous spectrum $\sigma_{\mathrm{c}}(T_\omega)$, the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(T_\omega)$, and the pure point spectrum $\sigma_{\mathrm{pp}}(T_\omega)$ are far harder, since one has to prove measurability of the projection P_{c} (resp. P_{pp}, P_{ac}) onto the continuous (resp. pure point, absolute continuous) subspace. Note that these projections depend on ω. In our notation $\sigma_{\mathrm{pp}}(T)$ is the closure of the set $\varepsilon(T)$ of eigenvalues of T. Unlike this closure, the set $\varepsilon(T_\omega)$ varies very rapidly with ω. In fact, $\varepsilon(T_\omega)$ seems to be the only spectral part which depends on ω.

The almost sure spectrum Σ is sometimes a very rough information about T_ω, as we will see later on. For instance, the almost sure spectrum of some classes of ergodic operators consists of the whole positive real line $[0,\infty)$, cf. Theorem 5.1. Therefore, we now look deeper into the spectrum, namely at the density of states. The density of states measure $\mu(I)$ of an operator expresses how many states can be distributed among an energy interval I. But we are dealing with infinitely extended operators in $L_2(\mathbb{R}^m)$ such that in our situation "how many" has no meaning at all. If we restrict our system to a finite portion of space, however, there remain only states of finite multiplicity. Then an appropriate average in space leads to the density of states.

We start with a description of the necessary technique. For $L \in \mathbb{N}$ let $Q_L = \left(-\frac{L}{2}, \frac{L}{2}\right]^m$ denote a cube of side length L centered at the origin. To restrict a selfadjoint operator T acting in $L_2(\mathbb{R}^m)$ to the cube Q_L we have to add suitable boundary conditions on ∂Q_L, the boundary of Q_L. The most interesting are Dirichlet and Neumann boundary conditions because, in some sense, they are minimal and maximal, respectively. The Dirichlet (resp. Neumann) operator $T_{Q_L}^{\mathrm{D}}$ (resp. $T_{Q_L}^{\mathrm{N}}$) on the box Q_L associated with T is given by $T_{Q_L}^{\mathrm{D/N}} = T\restriction_{L_2(Q_L)}$ with Dirichlet (resp. Neumann) boundary conditions on ∂Q_L. More precisely, $T_{Q_L}^{\mathrm{D}}$ is the unique selfadjoint operator associated with the quadratic form $\mathbf{t}^{\mathrm{D}}$ defined by the same formula (2.5) as $\mathbf{t}$ but with form domain $\mathcal{H}_\circ^1(Q_L)$; in the Neumann case the quadratic form $\mathbf{t}^{\mathrm{N}}$ has form domain $\mathcal{H}^1(Q_L)$. Here $\mathcal{H}_\circ^1(U)$ denotes the closure of $C_c^\infty(U)$, the space of smooth functions of any order defined on U with compact support, with respect to $\|\cdot\|_1$.

Definition 2.4 Let T be selfadjoint and let $\mathcal{N}_{Q_L}^{\mathrm{D/N}}(I)$ denote the number of eigenvalues of $T_{Q_L}^{\mathrm{D/N}}$ within an open interval I. We call

$$\mu(I) = \lim_{L\to\infty} \frac{1}{|Q_L|} \mathcal{N}_{Q_L}^{\mathrm{D}}(I) \tag{2.9}$$

the *density of states measure* of T, provided that the limit exists. Here $|\cdots|$ is the Lebesgue measure.

The reason for using the Dirichlet operator instead of the Neumann operator in the definition above is that it makes no difference in the case of both periodic and ergodic operators (for a proof see [Ki2] and references therein), the only operators we are going to deal with. Moreover, in both cases the limit (2.9) exists, at least if we average in the ergodic case.

Theorem 2.5 (Pastur) *Let $\{T_\omega\}$ be ergodic as above. Then for an open interval I the limit*

$$\mu(I) = \lim_{L\to\infty} \frac{1}{|Q_L|}\mathbb{E}\big[\mathcal{N}^{\mathrm{D}}_{Q_L}(I;T_\omega)\big]$$

exists, where $\mathbb{E}[\cdot]$ denotes the expectation with respect to P. In addition, the density of states measure $\mu(\cdot;T_\omega)$ exists P-a.s. and is P-a.s. independent of ω, i.e.

$$P\Big(\mu(I;T_\omega) = \mu(I)\Big) = 1,$$

for all $I = (a,b)$ with a and b continuity points of $\mu\big((a,b)\big)$.

We call μ the *density of states measure* of the ergodic family $\{T_\omega\}$. In most cases it is easier to investigate the behaviour of the distribution function

$$\rho(E) = \mu\big((-\infty,E)\big), \qquad E \in \mathbb{R}, \tag{2.10}$$

of the density of states measure instead of μ itself. ρ is called the *integrated density of states*. The almost sure spectrum Σ consists exactly of the growth points of ρ, or, in other words, $\Sigma = \operatorname{supp}\mu$. Here $\operatorname{supp}\mu$ denotes the support of the measure μ, i.e. $\operatorname{supp}\mu$ is the set of numbers $E \in \mathbb{R}$ for which any neighborhood U of E has positive measure $\mu(U) > 0$. Analogous to Eqn. (2.10), we will use $\mathcal{N}^{\mathrm{D/N}}_{Q_L}(E) = \mathcal{N}^{\mathrm{D/N}}_{Q_L}\big((-\infty,E)\big)$ as abbreviation.

Theorems 2.3 and 2.5 justify just to speak of spectral properties of ergodic operators instead of taking each realization ω into account. Those ω's that do not exhibit these almost sure spectrum and density of states are regarded as exceptions and will be omitted in our further considerations. So sometimes we drop the formulation "almost surely" in the sequel.

The proof of Theorem 2.5 gives the following easy conclusion.

Corollary 2.6 *For T_ω as in Theorem 2.5*

$$\frac{1}{|Q_L|}\mathbb{E}\big[\mathcal{N}^{\mathrm{D}}_{Q_L}(E;T_\omega)\big] =: \rho^{\mathrm{D}}_L(E) \le \rho(E) \le \rho^{\mathrm{N}}_L(E) := \frac{1}{|Q_L|}\mathbb{E}\big[\mathcal{N}^{\mathrm{N}}_{Q_L}(E;T_\omega)\big] \tag{2.11}$$

holds for all $L \in \mathbb{N}$ and $E \in \mathbb{R}$. Furthermore, $\rho^{\mathrm{D}}_L(E)$ (resp. $\rho^{\mathrm{N}}_L(E)$) increases (resp. decreases) monotonically as $L \to \infty$.

This relies on the usual Dirichlet-Neumann bracketing (cf. [RS4]). Since both the LHS and RHS of the enclosure (2.11) converge to $\rho(E)$, Corollary 2.6 supplies our main tool in investigating the density of states.

Our next goal is to approximate the spectrum and the density of states by periodic realizations, because we have a quite good knowledge of periodic models (see Chapter 4). The density of states is defined via approximation by operators on large boxes, and, since these operators are approximated by their periodic continuations themself, the definition of the density of states provides periodic approximations rather automatically.

For a similar determination of the spectrum we first have to make precise our random models.

3 Random models

In this chapter we will specify the random models employed here in detail. As mentioned in the introduction, our goal is to describe random divergence type operators $\nabla^* a_\omega(x)\nabla$ in $L_2(\mathbb{R}^m)$ with high contrast coefficients. More precisely, fix some large $\lambda > 1$ and take $\tilde{a}_\lambda \equiv \lambda$ as base medium occupying the whole space $\mathbb{R}^m$. Later on we will treat λ as a parameter and send λ to infinity. To generate a high contrast we shall perturb $\tilde{a}_\lambda$ by areas where the coefficients take smaller values than λ. For simplicity we will only deal with a two component medium, thus we assume the coefficients to take only two values, say 1 and λ. Therefore it remains to model the shape of the perturbing area.

We think of "grains" perturbing the base medium, hence for our models it is enough to (randomly) choose grains $M_1, M_2, \ldots$ and to consider coefficients of the form

$$a_\lambda(x) = \begin{cases} 1 & \text{if } x \in M_1 \cup M_2 \cup \ldots, \\ \lambda & \text{otherwise,} \end{cases} \tag{3.1}$$

and operators formally given by

$$T_\lambda = \nabla^* a_\lambda(x)\nabla \quad \text{in } L_2(\mathbb{R}^m). \tag{3.2}$$

Clearly, we have to take care of a_λ to be measurable (uniform ellipticity is obvious) for Eqn. (3.2) to be well defined. Moreover, the grains should be mutually disjoint, compact, and connected. Some of these demands are technically not necessary but this is what one usually would expect how "grains" are to behave.

For a technical reason (see Chapter 3.4) we make a general assumption on all grains that occur in our models.

Assumption 3.1 (Regularity assumption) *For the rest of this work we assume that all grains $M = \overline{M}$ occuring in our models satisfy*

$$\tilde{\mathcal{H}}^1_\circ(M) = \mathcal{H}^1_\circ(M^{\text{int}}), \tag{3.3}$$

where M^{int} denotes the interior of M,

$$\tilde{\mathcal{H}}^1_\circ(M) = \left\{ u \in \mathcal{H}^1(\mathbb{R}^m) \;\middle|\; u(x) = 0 \text{ a.e. in } \mathbb{R}^m \setminus M \right\},$$

and $\mathcal{H}^1_\circ(M^{\text{int}})$ the usual Sobolev space of the open set M^{int}. For example, if M satisfies the segment condition then (3.3) holds (cf. [HZh]).

Summarizing, and to take stochastic notation into account, all models consist in constructing $M(\omega)$, the union of perturbing grains. Thus $M(\omega)$ is always given by

$$M(\omega) = \bigcup_{j \in J} M_j(\omega),$$

where J denotes a suitable index set and $M_j(\omega)$ the individual grains. The random coefficients are then given by Eqn. (3.1) with $M_1 \cup M_2 \cup \ldots$ replaced by $M(\omega)$.

Intuitively there are two different kinds of randomness in modelling random grains: spatial and grain-shape randomness.

3.1 Spatial distribution (model M1)

The perturbations consist of grains of one shape Ξ, however, the grains are randomly distributed in space, in a way such that they do not touch each other. We say the grains are *spatially distributed*. More precisely, let Ξ be a closed subset of $\mathbb{R}^m$ that is contained in the closed ball of radius R centered at the origin, for R fixed, i.e. $\Xi \subset B(R)$. Assume that both Ξ^{int}, the interior of Ξ, and Ξ^c, the complement of Ξ, are connected. To distribute copies of this *primary grain* Ξ in $\mathbb{R}^m$, it is clearly enough to do so for the centres x_n of the enclosing balls $B(R) \supset \Xi$. Such random point processes are well known in stochastic geometry, we refer to [StSt, SKM] for details.

Let $\{x_n(\omega)\}_{n\in\mathbb{N}}$ denote a stationary Poisson point process in $\mathbb{R}^m$ with intensity $\kappa > 0$, i.e. $\Phi(\omega) = \{x_n(\omega) \mid n \in \mathbb{N}\}$ is simple and locally finite for all $\omega \in \Omega$, and the number of points of Φ in a bounded Borel set U has a Poisson distribution of mean $\kappa|U|$,

$$P\Big(\#\big(\Phi \cap U\big) = n\Big) = \frac{\kappa^n |U|^n}{n!}\, e^{-\kappa|U|}$$

(this justifies the name "Poisson"). Moreover, if we denote by $\#\big(\Phi \cap U\big)$ the number of points of Φ lying in U, then

$$\mathbb{E}\big[\#\big(\Phi \cap U\big)\big] = \kappa \cdot |U| \tag{3.4}$$

explains why we call κ *intensity* of Φ. The notation $\Phi = \{x_n \mid n \in \mathbb{N}\}$ might suggest that there is a natural order or enumeration of the points, but this is not the case. It can be easily seen that Φ is ergodic: for disjoint sets $U_1, U_2 \subset \mathbb{R}^m$ the random variables $\#\big(\Phi \cap U_k\big)$ are known to be independent, so that Φ is mixing (see Eqn. (2.3)), and therefore ergodic.

If we would take $\{x_n\}$ as centres of balls B_n with Radius R then the non-intersecting condition $B_n \cap B_k = \varnothing$ for $n \neq k$ is not guaranteed. Thus $(\Xi + x_n) \cap (\Xi + x_k) \neq \varnothing$ may happen (see Fig. 3.1).

A suitable refinement is performed by so called *hard-core processes*. These hard-core processes are special stationary point processes in which the constituent points are forbidden to lie closer together than a certain minimum distance, the *hard-core distance*. The points satisfy

$$\left|x_n(\omega) - x_k(\omega)\right| \geq h > 2R, \qquad \text{for } n \neq k, \quad n,k \in \mathbb{N}, \tag{3.5}$$

independent of ω, where $h > 0$ is the hard-core distance (see Fig. 3.2). For a construction procedure starting with a (simple) stationary Poisson process see [StSt, SKM]. Another approach leading to hard-core models uses stationary Gibbs point processes. For an introduction to Gibbs processes see [SKM]. Here we are only interested in properties (3.4) and (3.5) combined with the fact that hard-core processes are ergodic with respect to shifts τ_x, $x \in \mathbb{R}^m$ (cf. [St3, NZ2, NZ1, SKM]).

To illustrate ergodicity of hard-core processes we look at the mixing property (see Eqn. (2.3)). Consider cylinder sets A_1 and A_2, and suppose that their supporting points (the number of these points is finite!) are contained in compact sets $K_1, K_2 \subset \mathbb{R}^m$, respectively. Generally, for any $x \in \mathbb{R}^m$, A_1 and $\tau_x^{-1}A_2$ are not independent, so that $P(A_1 \cap \tau_x^{-1}A_2) \neq P(A_1)P(A_2)$. But the mutual influence should vanish if $|x| \to \infty$ and so the distance between K_1 and K_2 grows,

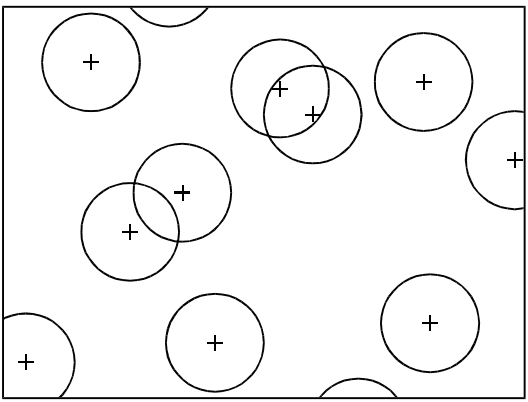


Figure 3.1: Simulation of a Poisson process with possibly intersecting balls of radius R centered at each point. Such models are known as *Boolean models.*

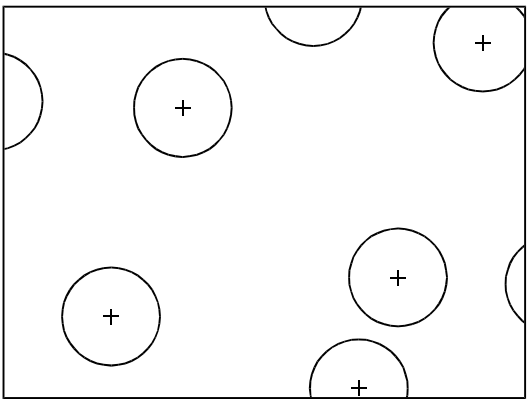


Figure 3.2: Simulation of a hard-core process with non-intersecting balls of radius $R < h/2$ centered at each point, and hard-core distance h.

i.e. $P(A_1 \cap \tau_x^{-1} A_2) \to P(A_1)P(A_2)$. That is what we called mixing, and so the process is ergodic.

In case of a construction procedure using Gibbs processes we can even show ergodicity directly. The main idea of stationary Gibbs processes is to start with a stationary Poisson point process, and then to weight the probability measure to obtain a new distribution. This concept allows to exclude certain configurations, for instance, those that do not respect a given hard-core distance. A fundamental characteristic of a Gibbs process is its so called *local energy function* E which gives via e^{-E} essentially the weight function mentioned above. We assume that E is of the form

$$E(x,\varphi) = \gamma + \sum_{y\in\varphi} \theta\big(|x-y|\big), \tag{3.6}$$

where γ is a constant (named chemical activity), and $\theta : [0,\infty) \to (-\infty,\infty]$ denotes the *pair potential*. Here $\varphi = \{x_n \mid n \in \mathbb{N}\}$ is a configuration of points in $\mathbb{R}^m$. For given x and φ the quantity $E(x,\varphi)$ can be interpreted as the energy needed to add the point x to φ. Gibbs processes first arose in statistical physics, hence the term. Hard-core models with hard-core distance h can be obtained if we set

$$\theta(d) = \begin{cases} \infty & \text{for } d \le h \\ 0 & \text{for } d > h, \end{cases} \tag{3.7}$$

since there cannot survive configurations with at least two points of distance

less or equal h. If the *interaction radius*

$$\zeta = \inf\{d \in \mathbb{R} \mid \theta(\rho) = 0 \text{ for all } \rho > d\}$$

is finite, then we have $P(A_1 \cap \tau_x^{-1} A_2) = P(A_1)P(A_2)$ with notation as above, provided $|x|$ is so large that the distance between K_1 and $\tau_x^{-1} K_2$ is greater than ζ. As $\zeta = h < \infty$ in our case, we conclude that the hard-core process is ergodic.

A further property of Gibbsian processes $\Phi(\omega)$ is that for any bounded Borel set $U \subset \mathbb{R}^m$ the probability that U contains no points of $\Phi(\omega)$ is positive,

$$P\{\Phi \cap U = \varnothing\} > 0. \tag{3.8}$$

This follows from a well known property of Gibbs point processes (see [Kal]), and Eqn. (3.8) turns out to be important for our further approach.

Finally, we define **model M1** to be

$$M(\omega) = \bigcup_{n \in \mathbb{N}} M_n(\omega), \quad M_n(\omega) = \Xi + x_n(\omega), \tag{M1}$$

with operators formally given by

$$T_{\omega,\lambda} = \nabla^* a_{\omega,\lambda} \nabla, \quad a_{\omega,\lambda}(x) = \begin{cases} 1 & \text{if } x \in M(\omega), \\ \lambda & \text{if } x \in \mathbb{R}^m \setminus M(\omega), \end{cases} \quad \lambda \geq 1, \tag{3.9}$$

where $\{x_n(\omega)\}$ denotes a Gibbs hard-core process in $\mathbb{R}^m$ with hard-core distance $h > 2R$ and intensity $\kappa > 0$. Of course, $T_{\omega,\lambda}$ is a random operator, i.e. $\omega \mapsto T_{\omega,\lambda}$ is weakly measurable (see Appendix A.1). The ergodicity of the hard-core process clearly implies that $T_{\omega,\lambda}$ is ergodic (with respect to $\mathbb{R}^m$-translations).

3.2 Grain-shape distribution (model M2)

As mentioned above there is another natural modelling of random grains, namely variation of the shape and size of grains. For a first investigation we do not want to consider random shapes together with spatial distributions, hence we restrict to a periodic lattice and pin randomly distributed grains at the lattice points.

To this end consider a distribution Γ of sets $\Xi \subset \mathbb{R}^m$ such that

$$\begin{aligned} &\Xi(\omega) \text{ is closed}, \quad \Xi(\omega)^{\text{int}}, \Xi(\omega)^c \text{ are connected}, \\ &\text{and} \quad \Xi(\omega) \subset V \subset Q_1^{\text{int}} \end{aligned} \tag{3.10}$$

hold with probability one, where V is some fixed compact set. We now fix at each lattice point such a random grain, i.e. we denote by P the product measure $\bigotimes_{i\in\mathbb{Z}^m}\Gamma$. Then **model M2** is defined as

$$M(\omega)=\bigcup_{i\in\mathbb{Z}^m} M_i(\omega),\quad M_i(\omega)=\Xi_i(\omega)+i, \tag{M2}$$

with operators formally given as in Eqn. (3.9). Here Ξ_i are independent identically distributed copies of the random set Ξ. Similarly to Section 2.2, independence implies that $T_{\omega,\lambda}$ is ergodic (with respect to $\mathbb{Z}^m$-translations). An example is shown in Fig. 3.3.

There are quite simple examples of such distributions Γ which, as we will see later, lead to an exact determination of the almost sure spectrum Σ_λ (see Theorem 5.2). The random grains are given by (random) scalings of a fixed (non-random) grain. Let $r(\omega)$ be a random variable with distribution Γ, and let Ξ be as in Eqn. (3.10) with $\Xi(\omega)$ replaced by Ξ. Moreover, assume that Ξ is star-shaped and has the origin as midpoint. Using the scaling operator

$$S_\eta:\mathbb{R}^m\to\mathbb{R}^m,\quad S_\eta x=\eta x,\qquad \text{for } \eta>0, \tag{3.11}$$

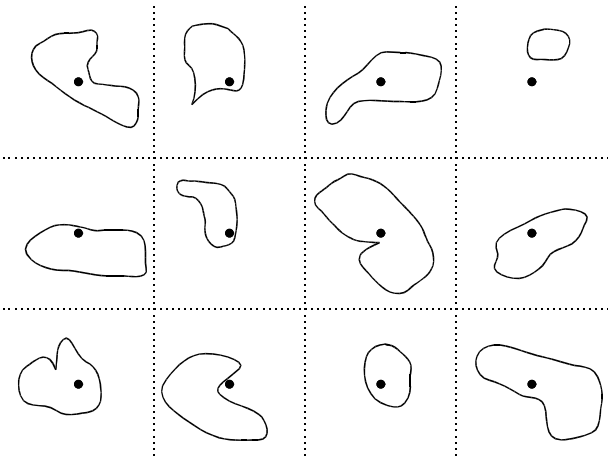


Figure 3.3: Simulation of model M2, i.e. a lattice point process with random grains $\Xi_i(\omega)$ independent identically distributed at each lattice point $i\in\mathbb{Z}^m$ (denoted by the bullets $\bullet$). The dotted lines separate the cubes Q_1+i and, therefore, the grains $\Xi_i(\omega)+i$, too.

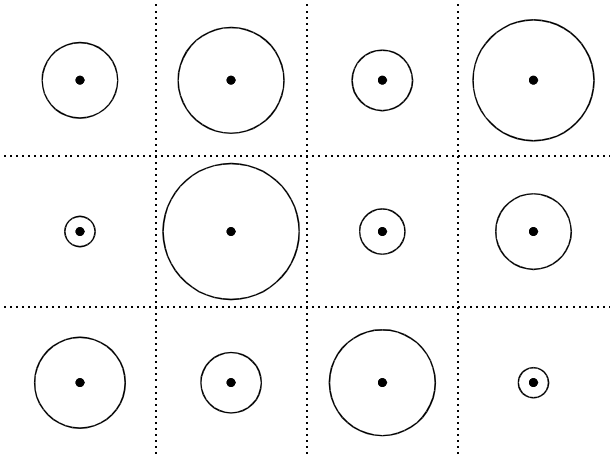


Figure 3.4: Simulation of model M2', i.e. a lattice point process with balls $B\big(r_i(\omega)/2\big)$ of random radius independent identically distributed at each lattice point $i \in \mathbb{Z}^m$ (denoted by the bullets $\bullet$).

we define

$$\Xi_i(\omega) = S_{r_i(\omega)}\Xi, \qquad i \in \mathbb{Z}^m, \tag{3.12}$$

where r_i are independent copies of the random variable r. Note that we have $S_{\eta_1}\Xi \subset S_{\eta_2}\Xi$ if $\eta_1 < \eta_2$. To guarantee condition (3.10) assume $\operatorname{supp}\Gamma \subset [R_{\min}, R_{\max}]$ with $0 < R_{\min} < R_{\max} \leq 1$.

Then **model M2'** is given by

$$M(\omega) = \bigcup_{i\in\mathbb{Z}^m} M_i(\omega), \quad M_i(\omega) = \Xi_i(\omega) + i. \tag{M2'}$$

A sample with $\Xi = B(1/2)$ is shown in Fig. 3.4.

3.3 Soft-core models (model M3)

In the two previous sections we have strictly distinguished between spatial and grain-shape randomness. Of course, both models, M1 and M2, can be combined leading to a further model M3. The reason for this temporary demarcation lies in the obtained spectral properties. If we put models M1 and M2 together, then the combined model partially carries some spectral properties

of **M**1 together with some properties of M2. On the other hand, some properties of M1 or M2 disappear. Summarizing, by creating a new model obtained form M1 together with M2 some kind of spectral properties are exhibited if at least one model, either M1 or M2, has this property, while other properties are only present, if both models, M1 and M2, have this particular property.

Model M3 should describe grains of variable size (and/or shape) which are randomly distributed in space. A first naive model can be obtained from hard-core models as in M1. Similar to model M2 we replace the fixed primary grain Ξ by a random primary grain $\Xi(\omega)$. To preserve mutually non-intersecting grains, we have to bound the radius R of the enclosing ball $\Xi(\omega) \subset B(R)$ by the half hard-core distance $h/2$ form above. But in such a model there cannot exist clusters of small grains since the enclosing balls have to respect the hard-core distance. The lack of clustering can be removed if we soften the (global) hard-core distance to a "local hard-core distance", depending on the local ball radii. This yields so called soft-core models which are also ergodic point processes (cf. [St3]). We will not discuss soft-core processes in detail but refer to the appropriate literature [StSt, St1, St2, St3]. For a sample of a soft-core process see Fig. 3.5.

Like in the case of hard-core models, there is also an intensity $\kappa > 0$ for a soft-core model Φ,

$$\mathbb{E}\big[\#\big(\Phi \cap U\big)\big] = \kappa \cdot |U|.$$

Moreover, each $x_n \in \Phi$ carries a mark r_n such that

$$\big|x_n(\omega) - x_k(\omega)\big| > r_n(\omega) + r_k(\omega), \quad n \neq k,$$

where the marks r_n are distributed according to a mark distribution Λ. This probability measure describes the behaviour of a typical point of Φ, where "typical" is meant in notation of Palm distributions (cf. [SKM]). For instance, the mean radius is given by $\mathbb{E}_\Lambda[r_1]$.

For example, such a soft-core process can be obtained as Gibbs process by modifying the pair potential θ in Eqn. (3.7), we now set

$$\theta(d, r_1, r_2) = \begin{cases} \infty & \text{if } d \leq r_1 + r_2 \\ 0 & \text{otherwise,} \end{cases}$$

i.e. the pair potential θ is infinite if the ball with radius r_1 centered at the origin and the ball with radius r_2 centered at a point with distance d from the origin do intersect. As for the energy function E, we have to take the marks (the radii) of the points into account. Let $\langle x, r\rangle \in \mathbb{R}^m \times (0,\infty)$ be a marked point, then define

$$E\big(\langle x, r\rangle, \varphi\big) = \gamma + \sum_{\langle y, r(y)\rangle \in \varphi} \theta\big(|x-y|, r, r(y)\big),$$

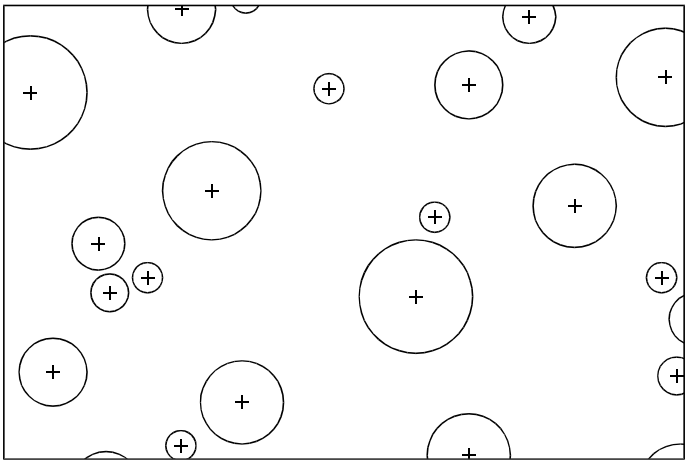


Figure 3.5: Simulation of a soft-core process with non-intersecting balls of random radius centered at each point.

where $r(y)$ denotes the mark (positive radius) of a point y of the configuration of marked points φ. In other words we have

$$E\big(\langle x,r\rangle,\varphi\big) = \begin{cases} \infty & \text{if for at least one } \langle y,r(y)\rangle \in \varphi \text{ the ball with radius } r(y) \text{ centered at } y \text{ intersects the ball with radius } r \text{ centered at } x \\ \gamma & \text{otherwise,} \end{cases}$$

replacing Eqn. (3.6). Now suppose the mark distribution Λ to have support in a compact interval $[0,\zeta/2]$. Thus two points $\langle y_i,r_i\rangle \in \Phi$ do not influence each other whenever $|y_1-y_2|>\zeta$. This justifies to call ζ the (finite) interaction radius. Applying the same arguments as above we obtain ergodicity of such soft-core processes. Furthermore, as in the case of hard-core processes, the same argument implies that Eqn. (3.8) holds also for Gibbs soft-core processes, i.e. any bounded Borel set contains no points of the point process with positive probability.

To put random grains Ξ_n into the balls $B(r_n)$ we use the scaling operators, introduced in Eqn. (3.11), and choose a random primary grain $\Xi \subset B(1)$, contained in the unit ball, with distribution Γ. With this notation we have $S_\eta \Xi \subset B(\eta)$.

Now let us turn back to our grain models. Let Φ be a Gibbs soft-core process as described above with mark distribution Λ such that $\operatorname{supp}\Lambda \subset [0,R]$,

for finite R. Moreover, denote by $\Xi_n(\omega)$ independent identically distributed copies of the random primary grain $\Xi(\omega)$. Then define **model M3** as

$$M(\omega) = \bigcup_{n\in\mathbb{N}} M_n(\omega), \quad M_n(\omega) = S_{r_n(\omega)}\Xi_n(\omega) + x_n(\omega), \qquad \textbf{(M3)}$$

with operators formally given by Eqn. (3.9). Again, $T_{\omega,\lambda}$ is ergodic (with respect to $\mathbb{R}^m$-translations). Here the probability measure P is the product measure of the distribution P_Φ of the soft-core process (which includes Λ in some sense) and the grain shape distribution $\bigotimes_{n\in\mathbb{N}}\Gamma$.

For a simple example take $\Xi = B(1)$ (non-random) which leads to individual grains

$$M_n(\omega) = B\big(r_n(\omega)\big) + x_n(\omega).$$

Fig. 3.5 shows a sample of this kind of model.

3.4 The limit operators

Once we have defined our models and, therefore, our operators $T_{\omega,\lambda}$ with parameter $\lambda \geq 1$, the first question that arises is wether there is a limit operator, say $T_{\omega,\infty}$, in some sense, as $\lambda \to \infty$.

For fixed ω, in all our models the operators $T_{\omega,\lambda}$ are monotonically increasing with respect to λ. Thus by monotone convergence of quadratic forms (cf. [Wei, Si, RS1]) there is a selfadjoint operator $T_{\omega,\infty}$, living in a possibly smaller Hilbert space, such that $T_{\omega,\lambda} \to T_{\omega,\infty}$ in strong resolvent sense, as $\lambda \to \infty$. The form domain $D(\mathbf{t}_{\omega,\infty})$ consists of all $u \in \mathcal{H}^1(\mathbb{R}^m)$ with $\sup_{\lambda>1} \mathbf{t}_{\omega,\lambda}[u] < \infty$, and $T_{\omega,\infty}$ lives in the Hilbert space $\tilde{\mathcal{H}}$ given by the closure of $D(\mathbf{t}_{\omega,\infty})$ in $L_2(\mathbb{R}^m)$. As the complement of $M(\omega)$ is open and connected in any case, we immediately get

$$D(\mathbf{t}_{\omega,\infty}) = \big\{ u \in \mathcal{H}^1(\mathbb{R}^m) \;\big|\; u(x) = 0 \text{ a.e. in } M(\omega)^c \big\} = \tilde{\mathcal{H}}^1_\circ\big(M(\omega)\big).$$

We refer to [HL1] for details. Thus, using Assumption 3.1, $D(\mathbf{t}_{\omega,\infty}) = \mathcal{H}^1_\circ(M(\omega)^{\text{int}})$, and we obtain

$$T_{\omega,\infty} = -\Delta_{M(\omega)} = \bigoplus_{j\in J} -\Delta_{M_j(\omega)}.$$

Here and in the sequel we denote, for simplicity, by $-\Delta_G$ the usual Dirichlet Laplacian on the open set $G^{\text{int}} \subset \mathbb{R}^m$ (we drop the superscript $^{\text{int}}$).

To summarize, under our assumptions we have

$$T_{\omega,\lambda} \longrightarrow \bigoplus_{j\in J} -\Delta_{M_j(\omega)} \quad \text{in s.r.s.,} \quad \text{as } \lambda \to \infty,$$

for all ω in all models M1–M3. Note that under fairly weak assumptions on the grains the limit operators on the RHS have discrete spectra.

3.5 Periodic approximation

In the present section we intend to characterize the spectrum Σ_λ of $T_{\omega,\lambda}$ for fixed λ more carefully. Using Floquet theory, the spectrum of periodic operators can be determined in quite an easy way, compared with random operators. Fortunately it turns out that the almost sure spectrum Σ_λ can be described via spectra of proper periodic operators.

To this end, consider the family of ergodic operators $T_{\omega,\lambda}$ according to one of the models M1–M3. Notice that we can take $L_{1,\mathrm{loc}}(\mathbb{R}^m)$ as state space. Clearly $L_{1,\mathrm{loc}}$ is a polish space; for instance, a metric generating the topology is given by

$$d(a,b) = \sum_{i\in\mathbb{Z}^m} \frac{1}{2^{|i|}} \min\left\{1, \|b-a\|_{L_1(Q_1+i)}\right\},$$

where $|\cdot|$ denotes the maximum norm on $\mathbb{R}^m$ and Q_1 the unit cube centered at the origin, as above. Furthermore, we set

$$\Omega_{\mathrm{per}} = \{\omega \in \Omega \mid a_\omega \text{ is periodic (with some finite period)}\},$$

the set of all periodic realizations.

The following Lemma states that the almost sure spectrum Σ_λ of the ergodic family $T_{\omega,\lambda}$ can be expressed by the spectra of all periodic realizations, i.e. by all $\omega \in \Omega_{\mathrm{per}} \cap \operatorname{supp} P$. This is a well know result for random Schrödiger operators (for a proof see [KM]). To prove the same result for divergence type operators one has to use different methods in parts compared to the Schrödinger case. For a model similar to Eqn. (2.7) a proof can be found in [FKl1] (Lemma 19). Our arguments presented here are a synthesis of [Ki2] and [FKl1].

Lemma 3.2 *Consider $T_{\omega,\lambda}$ as in models M1–M3, then*

$$\Sigma_\lambda = \overline{\bigcup_{\omega\in\Omega_{\mathrm{per}}\cap\operatorname{supp} P} \sigma(T_{\omega,\lambda})}.$$

Due to ergodicity Ω_{per} has measure zero or one. Thus it may happen that the almost sure spectrum is determined by a set of measure zero! As a preparation we need two Lemmas.

Lemma 3.3 *For $T_{\omega,\lambda}$ as in models M1–M3 suppose $\omega_n \to \omega_0$ in $L_{1,\mathrm{loc}}(\mathbb{R}^m)$. Then*

$$T_{\omega_n,\lambda} \longrightarrow T_{\omega_0,\lambda} \quad \textit{in s.r.s.},$$

and in particular

$$\sigma(T_{\omega_0,\lambda}) \subset \overline{\bigcup_{n\in\mathbb{N}} \sigma(T_{\omega_n,\lambda})}$$

holds.

We give a proof in Appendix A.2.

Lemma 3.4 *Let $T_{\omega,\lambda}$ be as in models M1–M3. If Ω_0 is dense in* $\operatorname{supp} P$ *then*

$$\Sigma_\lambda = \overline{\bigcup_{\omega\in\Omega_0} \sigma(T_{\omega,\lambda})}.$$

Proof. (1) Take some $\omega_0 \in \{\omega \in \operatorname{supp} P \mid \Sigma_\lambda = \sigma(T_{\omega,\lambda})\}$. Since $\Omega_0 \subset \operatorname{supp} P$ is dense we can find a sequence ω_n in Ω_0 with $\omega_n \to \omega_0$, thus Lemma 3.3 implies

$$\Sigma_\lambda = \sigma(T_{\omega_0,\lambda}) \subset \overline{\bigcup_{n\in\mathbb{N}} \sigma(T_{\omega_n,\lambda})} \subset \overline{\bigcup_{\omega\in\Omega_0} \sigma(T_{\omega,\lambda})}.$$

(2) As for the opposite inclusion, let $\omega_0 \in \operatorname{supp} P$ and define $U_n = \{\omega \in \Omega \mid d(\omega,\omega_0) < 1/n\}$, so $P(U_n) > 0$. By setting $\Omega_1 = \{\omega \in \operatorname{supp} P \mid \Sigma_\lambda = \sigma(T_{\omega,\lambda})\}$ we get $P(\Omega_1) = 1$ implying $P(U_n \cap \Omega_1) > 0$. Thus $U_n \cap \Omega_1$ is not empty and there is some $\omega_n \in U_n \cap \Omega_1$. As $\omega_n \to \omega_0$ we conclude

$$\sigma(T_{\omega_0,\lambda}) \subset \overline{\bigcup_{n\in\mathbb{N}} \sigma(T_{\omega_n,\lambda})} = \Sigma_\lambda,$$

again using Lemma 3.3. □

Now we return to Lemma 3.2.

Proof of Lemma 3.2. In view of Lemma 3.4 it remains to show that Ω_{per} is dense in $\operatorname{supp} P$. First we construct periodic realizations $\omega_n \in \Omega_{\mathrm{per}}$ approximating some $\omega_0 \in \operatorname{supp} P$.

For $n \in \mathbb{N}$ choose $L_n \in \mathbb{N}$ such that

$$\sum_{i \in \mathbb{Z}^m \cap Q_{L_n}^c} \frac{1}{2^{|i|}} < 1/n.$$

Then define $\omega_n \in \Omega_{\text{per}}$ by

$$\omega_n(q + L_n i) = \omega_0(q), \qquad i \in \mathbb{Z}^m, \tag{3.13}$$

where $q \in \mathbb{Z}^m \cap Q_{L_n}$ in case of model M2, and $q \in Q_{L_n}$ in case of model M1 or M3. We conclude $d(\omega_n, \omega_0) < 1/n$, such that $\omega_n \to \omega_0$, as $n \to \infty$.

To complete the proof we have to show that each ω_n belongs to $\operatorname{supp} P$. In case of model M2 this is immediately obvious since each $\omega_0(i) = \Xi_i(\omega_0)$ belongs to $\operatorname{supp} \Gamma$, the common distribution of the grain shape. As for model M1, we have to look more carefully at periodic continuations of $\omega_0 \restriction_{Q_{L_n}}$ as in Eqn. (3.13) because we have to avoid intersecting grains. Therefore, for given L_n and ω_0, drop all grains of ω_0 with center in Q_{L_n} that do intersect the boundary of Q_{L_n}, yielding $\tilde{\omega}_0$ with $d(\tilde{\omega}_0, \omega_0) < 1/n$, for L_n large enough. From Eqn. (3.8) one easily obtains $\tilde{\omega}_0 \in \operatorname{supp} P$, for instance, take $U = \{x \in \mathbb{R}^m \mid \operatorname{dist}(x, \partial Q_{L_n}) < h/2\}$ where h is the hard-core distance. Now, defining ω_n to be the periodic continuations of $\tilde{\omega}_0$ as in Eqn. (3.13) gives $\omega_n \in \operatorname{supp} P$ and $\omega_n \to \omega_0$, as required. For model M3 the same arguments apply. □

Together with Corollary 2.6 we are now well prepared to investigate both the spectrum and the density of states of random divergence type operators. Therefore, in the next chapter we look first at periodic operators.

4 The periodic case

We have seen in Chapters 2 and 3 that, in a certain sense, ergodic operators can be approximated by periodic operators. Thus we hope to reap some profit by investigating periodic operators that arise from restrictions of ergodic operators or from periodic realizations. These periodic operators should have a similar form as models M1–M3, i.e. M is supposed to be periodic,

$$M = \bigcup_{i \in \mathbb{Z}^m} (\Xi + i). \tag{4.1}$$

This particular class of periodic operators was studied by Hempel and Lienau in [HL1, HL2].

First note that by scaling it is sufficient to consider periodicity with respect to the $\mathbb{Z}^m$-lattice instead of $L\mathbb{Z}^m$-lattices, $L \in \mathbb{N}$, which are needed to approximate random operators (cf. Corollary 2.6 and Lemma 3.2). But in this case, the primary grain need not be connected, e.g. see Fig. 4.1. We start with a description of the operators used in [HL1]. Let $G \subset \mathbb{R}^m$ be open, connected, and periodic with respect to the $\mathbb{Z}^m$-lattice, and set $M = \mathbb{R}^m \setminus G$. Again, assume that M satisfies Assumption 3.1. Periodicity enables us to apply Floquet/Bloch theory, so it is enough to study restrictions of the periodic operators to the fundamental lattice cell $Q_1 = \left[-\frac{1}{2}, \frac{1}{2}\right)^m$. To this end, we denote

$$G_1 = G \cap Q_1, \qquad \Xi = M \cap Q_1,$$

and assume that Ξ has positive distance to the boundary of Q_1 (cf. Fig. 4.1). In fact, then Eqn. (4.1) holds. In analogy to the random case, define for $\lambda \geq 1$ selfadjoint operators T_λ^{per} in $L_2(\mathbb{R}^m)$ formally by

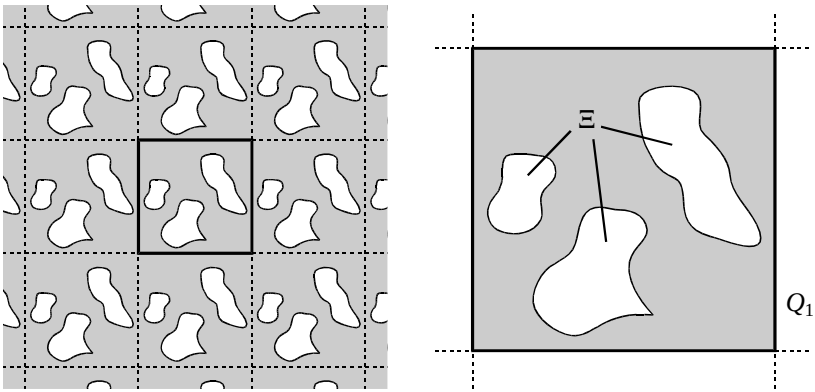
$$T_\lambda^{\mathrm{per}} = \nabla^* a_\lambda^{\mathrm{per}} \nabla, \qquad a_\lambda^{\mathrm{per}}(x) = \begin{cases} 1 & \text{if } x \in M, \\ \lambda & \text{if } x \in G. \end{cases} \tag{4.2}$$

To state the results given in [HL1, HL2] we need the following notation. Using Floquet theory, T_λ^{per} can be decomposed into fibers

$$T_\lambda^{\mathrm{per}} \cong (2\pi)^{-m} \int_{[-\pi,\pi)^m}^{\oplus} T_\lambda^{\mathrm{per}}(\vartheta)\, d\vartheta,$$

where $T_\lambda^{\mathrm{per}}(\vartheta)$ is the restriction of T_λ^{per} on Q_1 with ϑ-periodic boundary conditions. $E_{k,\lambda}(\vartheta)$ denotes the eigenvalues with numbering according to min-max, counting multiplicities. If we let

$$\alpha_{k,\lambda} = \min_{\vartheta} E_{k,\lambda}(\vartheta), \qquad \beta_{k,\lambda} = \max_{\vartheta} E_{k,\lambda}(\vartheta), \tag{4.3}$$



(a) (b)

Figure 4.1: Sample of a periodic operator: (a) A part of the periodic lattice, and (b) the fundamental lattice cell Q_1 and the fundamental (or primary) grain Ξ.

then the spectrum of T_λ^{per} is given by

$$\sigma(T_\lambda^{\mathrm{per}}) = \bigcup_{k\in\mathbb{N}} [\alpha_{k,\lambda}, \beta_{k,\lambda}]. \tag{4.4}$$

We call the interval $[\alpha_{k,\lambda}, \beta_{k,\lambda}]$ the *k-th spectral band*, and $(\beta_{k,\lambda}, \alpha_{k+1,\lambda})$, provided it is not empty, the *k-th spectral gap* of T_λ^{per}.

In the same way as for ϑ-periodic operators, we introduce Dirichlet resp. Neumann boundary conditions on ∂Q_1 (cf. Section 2.3) leading to operators $T_\lambda^{\mathrm{per,D}}$ resp. $T_\lambda^{\mathrm{per,N}}$ with eigenvalues $E_{k,\lambda}^{\mathrm{D}}$ and $E_{k,\lambda}^{\mathrm{N}}$, respectively. For all $k \in \mathbb{N}$, min-max implies

$$E_{k,\lambda}^{\mathrm{N}} \le E_{k,\lambda}(\vartheta) \le E_{k,\lambda}^{\mathrm{D}}, \quad \vartheta \in [-\pi,\pi)^m,$$

and so

$$\sigma(T_\lambda^{\mathrm{per}}) \subset \bigcup_{k\in\mathbb{N}} [E_{k,\lambda}^{\mathrm{N}}, E_{k,\lambda}^{\mathrm{D}}].$$

Moreover, $E_{k,\lambda}^{\mathrm{D}}$ and $E_{k,\lambda}^{\mathrm{N}}$ increase monotonically and converge as $\lambda \to \infty$,

$$\delta_k := \lim_{\lambda\to\infty} E_{k,\lambda}^{\mathrm{D}} = \sup_{\lambda>1} E_{k,\lambda}^{\mathrm{D}}, \qquad \nu_k := \lim_{\lambda\to\infty} E_{k,\lambda}^{\mathrm{N}} = \sup_{\lambda>1} E_{k,\lambda}^{\mathrm{N}}.$$

With this notation we can state the following result.

Theorem 4.1 (Thm. 2.4(a) in [HL1]) *The spectrum of T_λ^{per} converges to $\bigcup_k [\nu_k, \delta_k]$, as $\lambda \to \infty$, in the sense that*

$$\alpha_{k,\lambda} \to \nu_k, \quad \beta_{k,\lambda} \to \delta_k, \qquad \text{as } \lambda \to \infty.$$

Moreover, the limit eigenvalues interlace, i.e. $\nu_k \le \delta_k \le \nu_{k+1}$, for $k \in \mathbb{N}$.

Summarizing, Theorem 4.1 says that the spectrum of T_λ^{per} in any compact interval $[0,K]$ consists, up to $\varepsilon > 0$, of $\bigcup_k [\nu_k, \delta_k]$, for λ large, i.e.

$$\Big([0,K] \cap \bigcup_k \big[\nu_k, \delta_k - \varepsilon\big]\Big) \;\subset\; \Big([0,K] \cap \sigma(T_\lambda^{\mathrm{per}})\Big) \;\subset\; \Big([0,K] \cap \bigcup_k \big[\nu_k - \varepsilon, \delta_k\big]\Big) \tag{4.5}$$

for $\lambda \ge \Lambda(\varepsilon, K)$, taking the monotonicity of $E_{k,\lambda}^{\mathrm{D/N}}$ into account.

While the limit Neumann eigenvalues ν_k are not related to any Neumann problem, the δ_k's are real Dirichlet eigenvalues. Under the regularity conditions made above (see Assumption 3.1) we have

$$T_\lambda^{\mathrm{per}} \longrightarrow -\Delta_M = \bigoplus_{i \in \mathbb{Z}^m} -\Delta_{\Xi + i} \quad \text{in strong resolvent sense}, \quad \lambda \to \infty,$$

(cf. [HL1] and Section 3.4), and we see that the T_λ^{per} do not converge in norm resolvent sense since the limit spectrum may consist of non-degenerate bands, while $\sigma(-\Delta_M) = \{\delta_k \mid k \in \mathbb{N}\}$. Note that the δ_k's are exactly the eigenvalues (with finite multiplicity) of the Dirichlet Laplacian $-\Delta_\Xi$ on Ξ, and $T_\infty^{\mathrm{per,D}} = \lim T_\lambda^{\mathrm{per,D}}$ coincides with $-\Delta_\Xi$, and each $E \in \sigma(-\Delta_M)$ is an eigenvalue of infinite multiplicity.

Not only the spectrum but also the density of states is determined by the band functions $E_{k,\lambda}$ via

$$\mu_\lambda(I) = (2\pi)^{-m} \sum_{k \in \mathbb{N}} \big|\{\vartheta \mid E_{k,\lambda}(\vartheta) \in I\}\big|, \tag{4.6}$$

so there is also a limit result for the density of states. Here $|\cdots|$ denotes Lebesgue measure.

Theorem 4.2 (Thm. 2.4(b) in [HL1]) *The density of states measure $\mu_\lambda^{\mathrm{per}}$ associated with T_λ^{per} converges vaguely to the density of states measure $\mu_\infty^{\mathrm{per}}$ of $-\Delta_M$, given by the counting function of the set $\{\delta_k \mid k \in \mathbb{N}\}$. In detail, for an open interval $I \subset \mathbb{R}$ with endpoints not in the set $\{\delta_k \mid k \in \mathbb{N}\}$ we have*

$$\mu_\lambda^{\mathrm{per}}(I) \longrightarrow \mu_\infty^{\mathrm{per}}(I) = \sum_{\delta_k \in I} 1, \qquad \text{as } \lambda \to \infty.$$

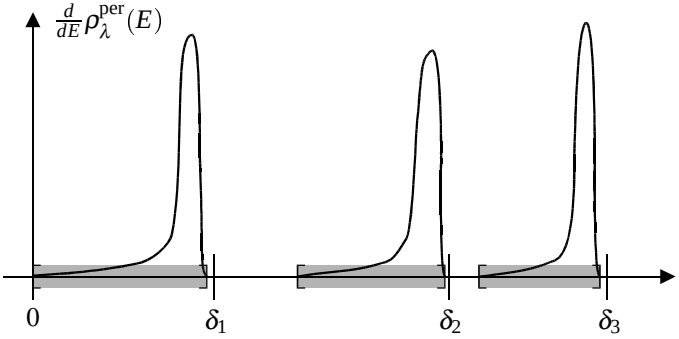


Figure 4.2: Typical density function of the integrated density of states $\rho_\lambda^{\mathrm{per}}$ in the periodic case.

The above theorem is a simple consequence of Eqn. (4.6) combined with the fact that all $E_{k,\lambda}(\vartheta)$ concentrate at δ_k, as $\lambda \to \infty$, except for $E_{k,\lambda}(0)$ that converges to ν_k.

We will now look more carefully at the limit behaviour of spectral bands and existence of gaps, for λ large. Obviously, Theorem 4.1 implies, provided there is strict interlacing

$$\nu_k < \delta_k < \nu_{k+1}, \tag{4.7}$$

that the k-th band of $\sigma(T_\lambda^{\mathrm{per}})$ does not collapse into a point, as $\lambda \to \infty$, while the k-th gap opens up, as $\lambda \to \infty$. Here the first result follows from the LHS of Eqn. (4.7), the second from the RHS. In [HL2] there is established the existence of grains Ξ such that (4.7) holds for infinitely many $k \in \mathbb{N}$. More precisely, the grains Ξ satisfying (4.7), for any k up to some arbitrary $K \in \mathbb{N}$, form a dense G_δ-set in a suitable grain-space, i.e. (4.7) is a generic property. Thus, generically, the density of states behaves, for λ large, as is shown in Fig. 4.2. The shaded area of the real line indicates the spectrum.

For later use notice that there exist grains such that any number of gaps in $\sigma(T_\lambda^{\mathrm{per}})$ will open up, as $\lambda \to \infty$.

5 Location of the spectrum and convergence of the density of states

In this chapter we describe the location of the almost sure spectrum Σ_λ and the behaviour of the almost sure density of states μ_λ of models M1–M3 in more detail, as $\lambda \to \infty$. Surprisingly, in most cases, our random operators behave almost like periodic operators, discussed in the previous chapter. We start with the spectrum.

5.1 Determination of the spectrum

First consider model M1, spatial distribution. There is just one primary grain, like in the periodic case, which is randomly distributed in space. Hence one could conjecture that the spectrum Σ_λ has also a band-gap structure, and that there exist spectral gaps, at least for λ large. But quite the contrary is true.

Theorem 5.1 *Consider model M1 or model M3. For all $\lambda \geq 1$ the a.s. spectrum Σ_λ of $T_{\omega,\lambda}$ coincides with the whole positive real line. More precisely,*

$$\sigma(T_{\omega,\lambda}) \overset{\text{a.s.}}{=} \Sigma_\lambda = [0,\infty), \qquad \text{all } \lambda \geq 1,$$

holds.

Proof. For the moment assume that $\tilde{\omega} = \tilde{a} \equiv \lambda$ belongs to $\operatorname{supp} P$. Then Lemma 3.2 immediately implies $\sigma(T_{\tilde{\omega},\lambda}) \subset \Sigma_\lambda$, and since $\sigma(T_{\tilde{\omega},\lambda}) = \sigma(-\lambda\Delta) = [0,\infty)$ the claimed result follows.

It remains to show that $\tilde{\omega} \in \operatorname{supp} P$. For this take $\omega_n \in \operatorname{supp} P$ such that $M(\omega_n) \cap Q_n = \varnothing$. Such realizations exist because using Eqn. (3.8) we see that $\{\omega \mid M(\omega) \cap Q_n = \varnothing\}$ has positive measure, for all n. With this choice we get $\omega_n \to \tilde{\omega}$ yielding $\tilde{\omega} \in \operatorname{supp} P$. □

This result makes it clear that it is more fruitful to investigate the density of states instead of the spectrum itself. We will look at the density of states in the next section.

In the case of model M2 the lattice $\mathbb{Z}^m$, which the model is based on, is strong enough to carry over the band-gap structure from the periodic case. Unfortunately, without further assumptions, Lemma 3.2 cannot be improved in

the general case. So we first investigate the more specific model M2', the lattice model with randomly scaled copies of a fixed (non-random) grain $\Xi \subset Q_1$.

It turns out that the spectrum Σ_λ is completely determined by the spectra of $T_{R,\lambda}$, the $\mathbb{Z}^m$-periodic operator with

$$M(R) = \bigcup_{i\in\mathbb{Z}^m} \big(S_R\Xi + i\big), \qquad R \in \operatorname{supp}\Gamma. \tag{5.1}$$

Under some additional assumption on $\operatorname{supp}\Gamma$, we can even express Σ_λ in terms of only two spectra, namely of $T_{R_{\min},\lambda}$ and $T_{R_{\max},\lambda}$.

Theorem 5.2 *Consider model M2'. If* $\operatorname{supp}\Gamma = [R_{\min}, R_{\max}]$ *then*

$$\sigma(T_{\omega,\lambda}) \overset{\text{a.s.}}{=} \Sigma_\lambda = \bigcup_{k\in\mathbb{N}} \Big[\alpha_{k,\lambda}(R_{\max})\,,\, \beta_{k,\lambda}(R_{\min})\Big], \qquad \lambda \geq 1,$$

where $\alpha_{k,\lambda}(R_{\max})$ *and* $\beta_{k,\lambda}(R_{\min})$ *are the (lower resp. upper) band edges of the spectra of* $T_{R_{\max},\lambda}$ *and* $T_{R_{\min},\lambda}$*, respectively, cf. Eqns. (4.3) and (4.4).*

See Fig. 5.1 for an example. For the proof we need the following result.

Lemma 5.3 *Let* $\alpha_{k,\lambda}(R)$ *and* $\beta_{k,\lambda}(R)$ *denote the edges of the spectral bands of the periodic operator* $T_{R,\lambda}$ *given by Eqn. (5.1).*

Then, for fixed λ, $\alpha_{k,\lambda}(\cdot)$ *and* $\beta_{k,\lambda}(\cdot)$ *are continuous on* $[R_{\min}, R_{\max}]$.

Proof. Fix some $R \in (R_{\min}, R_{\max})$, and define $\Xi(\kappa) = S_{R+\kappa}\Xi$, for $\kappa \in I = (-\varepsilon, \varepsilon)$. Our idea is to apply a domain perturbation as in Kato's book [Kat, VII-§6.5]. Therefore, we need a group of diffeomorphisms $\Psi(\kappa) \in C^1(\mathbb{R}^m, \mathbb{R}^m)$

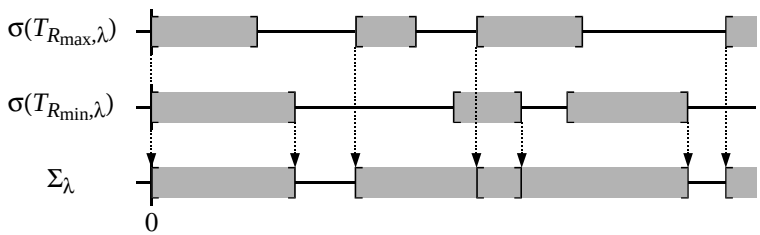


Figure 5.1: Almost sure spectrum Σ_λ in model M2'.

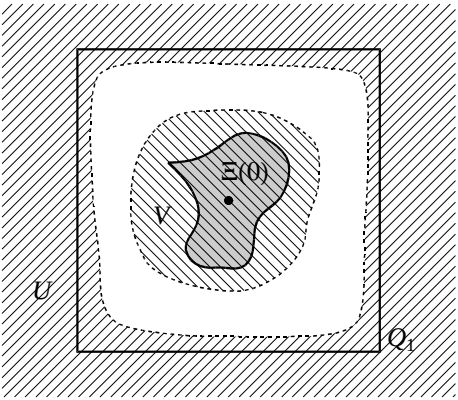


Figure 5.2: The sets U, V used in the proof of Lemma 5.3.

depending on $\kappa \in I$ in such a way that $\Psi(\kappa)$ transforms $\Xi(0) = S_R\Xi$ to $\Xi(\kappa) = S_{R+\kappa}\Xi$, and at the same time leaves Q_1 invariant.

To this end, let U, V be open subsets of $\mathbb{R}^m$ with $\operatorname{dist}(U,V) > 0$, $\overline{Q_1^c} \subset U$, and $\Xi(0) \subset V$ (cf. Fig. 5.2). Define

$$\tilde{\phi}(x) = \frac{\operatorname{dist}(x,U)}{\operatorname{dist}(x,U) + \operatorname{dist}(x,V)}\, x, \qquad x \in \mathbb{R}^m,$$

thus

$$\tilde{\phi} = \mathrm{id} \quad \text{on } V, \qquad \tilde{\phi} = 0 \quad \text{on } U, \tag{5.2}$$

and $\tilde{\phi}$ is continuous. Now smooth out $\tilde{\phi}$ to obtain $\phi \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ so that (5.2) holds for ϕ, too. Finally, with

$$\Psi(\kappa) : \mathbb{R}^m \to \mathbb{R}^m, \quad x \mapsto x + \kappa\phi(x), \qquad \kappa \in I,$$

we get the desired diffeomorphisms.

Denote by $a_{R+\kappa,\lambda}$ the $\mathbb{Z}^m$-periodic coefficients of $T_{R+\kappa,\lambda}$, and consider the eigenvalue problem

$$\nabla^* a_{R+\kappa,\lambda} \nabla u = E(\kappa) u \quad \text{on } Q_1 \tag{5.3}$$

with ϑ-periodic boundary conditions on ∂Q_1. Note that $a_{R+\kappa,\lambda} \circ \Psi(\kappa) = a_{R,\lambda}$. Via $\Psi(\kappa)$ we transform the eigenvalue problem (5.3), with eigenfunctions $u(y;\kappa)$, into a generalized eigenvalue problem on Q_1, with independent variable x and eigenfunctions $\hat{u}(x;\kappa)$, where the quadratic forms corresponding to

Eqns. VII-(6.39) and VII-(6.40) in [Kat] are given by

$$\begin{aligned}\mathbf{t}_\lambda(\kappa)[\hat{u}] &= \int_{Q_1} a_{R+\kappa,\lambda}(y)\left|\nabla u(y)\right|^2 dy \\ &= \int_{Q_1} a_{R,\lambda}(x)\left|\mathbf{J}(x;\kappa)^{-1}\cdot\nabla\hat{u}(x)\right|^2 J(x;\kappa)\,dx, \\ \mathbf{a}(\kappa)[\hat{u}] &= \int_{Q_1} |u(y)|^2\,dy = \int_{Q_1} |\hat{u}(x)|^2 J(x;\kappa)\,dx.\end{aligned} \tag{5.4}$$

Here we write $y = \Psi(x;\kappa)$, and $\mathbf{J}(x;\kappa) = \left(\delta_{kl} + \kappa\frac{\partial\phi_k}{\partial x_l}(x)\right)$ for the Jacobian of the transformation $\Psi(\kappa)$, and $J(x;\kappa) = \det\mathbf{J}(x;\kappa)$. Note that $\hat{u}$ satisfies ϑ-periodic boundary conditions on ∂Q_1 if and only if u does.

Now we can proceed as in [Kat]. It follows that the eigenvalues $E(\kappa)$ of (5.3) depend analytically on $\kappa \in I$, and so both the minimum $\alpha_{k,\lambda}(R+\kappa)$, taken over $\vartheta \in [-\pi,\pi)^m$, and the maximum $\beta_{k,\lambda}(R+\kappa)$ are clearly continuous in κ, proving the Lemma. □

Remark 5.4 (a) In the proof of Lemma 5.3, ϑ-periodic boundary conditions can be replaced by Dirichlet or Neumann boundary conditions, then leading to piecewise analytic eigenvalues $E^{\mathrm{D}}_{k,\lambda}(R)$ and $E^{\mathrm{N}}_{k,\lambda}(R)$ with respect to R. Note also that the proof just gives analytic functions $E_k(\cdot)$, $k \in \mathbb{N}$, that coincide each with, for instance, $E^{\mathrm{D}}_{k,\lambda}(\cdot)$, at some fixed R. However, the numbering of E_k may be inconsistent with the one obtained by min-max.

(b) In our situation we cannot use the usual resolvent equation technique,

$$\left\|(T_{R_1,\lambda}+1)^{-1} - (T_{R_2,\lambda}+1)^{-1}\right\| \le C\left\|a_{R_1,\lambda} - a_{R_2,\lambda}\right\|_\infty,$$

to derive continuity of the eigenvalues because of $\left\|a_{R_1,\lambda} - a_{R_2,\lambda}\right\|_\infty = \lambda - 1$, whenever $R_1 \neq R_2$. Moreover, the coefficients are not analytic themselves, i.e. $a_{R+\kappa,\lambda}$ cannot be expanded into a series $a_{R,\lambda} + \kappa a^{(1)}_\lambda + \kappa^2 a^{(2)}_\lambda + \cdots$. Thus, $T_{R+\kappa,\lambda}$ is not an analytic family of operators of type (B) in the sense of Kato (cf. [Kat]). But if $\tilde{T}_{R+\kappa,\lambda}$ denotes the operator corresponding to Eqn. (5.4), then we get an analytic family of type (B), and, furthermore, the eigenvalues of $\tilde{T}_{R+\kappa,\lambda}$ and the eigenvalues of $T_{R+\kappa,\lambda}$ are identical.

Now we are well prepared to prove Theorem 5.2.

Proof of Theorem 5.2. For $R \in [R_{\min}, R_{\max}]$ the operator $T_{R,\lambda}$ is periodic, so Lemma 3.2 implies

$$\Big(\bigcup_{R\in[R_{\min},R_{\max}]} \sigma(T_{R,\lambda}) \Big) \subset \Sigma_\lambda. \tag{5.5}$$

Conversely, if $\omega \in \Omega_{\mathrm{per}} \cap \operatorname{supp} P$ is periodic with respect to $L\mathbb{Z}^m$, for some $L \in \mathbb{N}$, then $M(R_{\min}) \subset M(\omega) \subset M(R_{\max})$ yields

$$a_{R_{\max},\lambda} \le a_{\omega,\lambda} \le a_{R_{\min},\lambda} \quad \text{on } \mathbb{R}^m, \tag{5.6}$$

and by min-max (all three operators are periodic!)

$$\sigma(T_{\omega,\lambda}) \subset \bigcup_{k\in\mathbb{N}} \Big[\alpha_{k,\lambda}(R_{\max}) \, , \, \beta_{k,\lambda}(R_{\min}) \Big],$$

for all periodic realizations $T_{\omega,\lambda}$. The RHS does not depend on ω, hence Lemma 3.2 again gives

$$\Sigma_\lambda \subset \bigcup_{k\in\mathbb{N}} \Big[\alpha_{k,\lambda}(R_{\max}) \, , \, \beta_{k,\lambda}(R_{\min}) \Big]. \tag{5.7}$$

It remains to show that the LHS of Eqn. (5.5) coincides with the RHS of Eqn. (5.7). By Lemma 5.3 the spectral edges of $\sigma(T_{R,\lambda})$ depend continuously on R, moreover, they are clearly monotonically decreasing, thus

$$\Big(\bigcup_{R\in[R_{\min},R_{\max}]} \sigma(T_{R,\lambda}) \Big) = \bigcup_{k\in\mathbb{N}} \Big[\alpha_{k,\lambda}(R_{\max}) \, , \, \beta_{k,\lambda}(R_{\min}) \Big],$$

proving the Theorem. □

Note that we would only get an inclusion "$\subset$" instead of an equality in Theorem 5.2 if Lemma 5.3 would not be true. The same applies to the case where $\operatorname{supp}\Gamma \subsetneq [R_{\min}, R_{\max}]$ is a strict inclusion.

There are some obvious consequences of Theorem 5.2.

Corollary 5.5 *Consider model M2'. Then the following holds.*

(a) In the definition of model M2' let Ξ be a grain such that the associated periodic operator (via Eqns. (4.1) and (4.2)) exhibits gaps in its spectrum, as $\lambda \to \infty$. (cf. end of Chapter 4)

If $|R_{\max} - R_{\min}|$ is small enough then Σ_λ has open gaps, for λ large.

(b) Assume $\operatorname{supp}\Gamma = [R_{\min}, R_{\max}]$. Then, analogous to Eqn. (4.5), for $\varepsilon > 0$ and $K > 0$ we have

$$\Big([0,K]\cap\bigcup_k\big[\nu_k(R_{\max}),\delta_k(R_{\min})-\varepsilon\big]\Big)$$
$$\subset\ \Big([0,K]\cap\Sigma_\lambda\Big)\ \subset\ \Big([0,K]\cap\bigcup_k\big[\nu_k(R_{\max})-\varepsilon,\delta_k(R_{\min})\big]\Big),$$

for $\lambda\geq\Lambda(\varepsilon,K)$.

Proof. For fixed R the periodic operator $T_{R,\lambda}$ has open gaps by assumption, for λ large, then, using Theorems 4.1 and 5.2, part (a) follows. Part (b) is clear from Theorem 4.1. □

For a general grain distribution Γ, i.e. model M2, one obtains a representation as in Theorem 5.2 using the above arguments if the following conditions (replacing mainly Lemma 5.3) are satisfied. First, the band edges of $\mathbb{Z}^m$-periodic operators $T_{\omega,\lambda}$ depend continuously on ω, the support of Γ is connected, and, finally, there is some monotonicity or some enclosure like Eqn. (5.6).

But, generally, these conditions are not satisfied, and we can only determine the almost sure spectrum via

$$\Sigma_\lambda=\overline{\bigcup_{\omega\in\Omega_{\rm per}\cap\operatorname{supp}P}\sigma(T_{\omega,\lambda})} \tag{5.8}$$

(cf. Lemma 3.2). We will briefly describe why the arguments used above break down in the general case of model M2. The starting point of a sharper determination of Σ_λ is given, as usual, by the set of all periodic realizations $T_{\omega,\lambda}$, $\omega\in\Omega_{\rm per}$, used in Eqn. (5.8). Therefore, fix some $\omega_0\in\Omega_{\rm per}$ such that $T_{\omega_0,\lambda}$ is periodic with respect to, say, the $L\mathbb{Z}^m$-lattice. $T_{\omega_0,\lambda}$ is completely determined by the set of L^m grains $\Xi_i(\omega_0)$, for $i\in Q_L\cap\mathbb{Z}^m$; thus we have to look only at the basic lattice cell $Q_L=\big[-\frac{L}{2},\frac{L}{2}\big)^m$. Our goal would be to obtain an enclosure of the spectral bands of $T_{\omega_0,\lambda}$ in terms of the spectral bands of $\mathbb{Z}^m$-periodic operators $T^{\rm per}_{\Xi,\lambda}$, for $\Xi\in\operatorname{supp}\Gamma$. As above, $T^{\rm per}_{\Xi,\lambda}$ denotes the operator with $\mathbb{Z}^m$-periodic grain area given by

$$M=\bigcup_{i\in\mathbb{Z}^m}\big(\Xi+i\big).$$

To this end, we had to compare $T_{\omega_0,\lambda}$ with some $T^{\rm per}_{\Xi,\lambda}$, but the periods are not identical. Hence we should consider $T^{\rm per}_{\Xi,\lambda}$ as $L\mathbb{Z}^m$-periodic operator, which obviously can be done without changing the spectral bands. At this point we

encounter a problem. There is no need that there exist grains, say $\Xi_{\min}$ and $\Xi_{\max}$, in $\operatorname{supp}\Gamma$ such that

$$\Xi_{\min} \subset \Xi_i(\omega_0) \subset \Xi_{\max}, \qquad \text{for all } i \in Q_L \cap \mathbb{Z}^m,$$

respectively

$$T^{\mathrm{per}}_{\Xi_{\max},\lambda} \leq T_{\omega_0,\lambda} \leq T^{\mathrm{per}}_{\Xi_{\min},\lambda}, \qquad \lambda > 1,$$

holds. The last inequality would lead to an appropriate band enclosure for $T_{\omega_0,\lambda}$ not depending on ω_0.

So all we can do is, for the sake of completeness, to state

Theorem 5.6 *Consider model M2.*

(a) We have

$$\sigma(T_{\omega,\lambda}) \overset{\text{a.s.}}{=} \Sigma_\lambda = \overline{\bigcup_{\omega \in \Omega_{\mathrm{per}} \cap \operatorname{supp} P} \sigma(T_{\omega,\lambda})}, \qquad \lambda \geq 1.$$

(b) Assume there exist a minimal and a maximal grain in $\operatorname{supp}\Gamma$*, in the sense that* $\Xi_{\min} \subset \Xi(\omega) \subset \Xi_{\max}$ *for all* $\omega \in \operatorname{supp}\Gamma$.

Then

$$\sigma(T_{\omega,\lambda}) \overset{\text{a.s.}}{=} \Sigma_\lambda \subset \bigcup_{k \in \mathbb{N}} \left[\alpha_{k,\lambda}(\Xi_{\min}) \, , \beta_{k,\lambda}(\Xi_{\max}) \right]$$

holds for all $\lambda \geq 1$. *Here* $\alpha_{k,\lambda}(\Xi_{\min})$ *resp.* $\beta_{k,\lambda}(\Xi_{\max})$ *denote the lower resp. upper band edges of the* $\mathbb{Z}^m$*-periodic operators* $T^{\mathrm{per}}_{\Xi_{\min},\lambda}$ *resp.* $T^{\mathrm{per}}_{\Xi_{\max},\lambda}$.

5.2 Concentration regions and convergence of the density of states

In all our models, the operators $T_{\omega,\lambda}$ converge in strong resolvent sense to the limit operator $-\Delta_{M(\omega)}$. But this convergence is too weak to ensure spectral convergence. For example, in the case of model M1, Theorem 5.1 says that $T_{\omega,\lambda}$ has spectrum equal to $[0,\infty)$ almost surely, for $\lambda < \infty$, while the limit spectrum consists exactly of the discrete Dirichlet eigenvalues of $-\Delta_\Xi$, the Dirichlet Laplacian on the primary grain.

In contrast, the convergence in strong resolvent sense yields that the density of states measure μ_λ converges to the density of states measure μ_∞ of the limit operator. For details see Theorem 5.9. Preparatory to proving this claim, we give a more general statement.

Lemma 5.7 *Suppose we are given selfadjoint operators S_n, S in $L_2(\mathbb{R}^m)$, S_n monotonically increasing or decreasing, and $S_n \to S$ in strong resolvent sense. If the integrated densities of states $\rho(\,\cdot\,;S_n), \rho(\,\cdot\,;S)$ exist, then the integrated density of states converges,*

$$\rho(E;S_n) \longrightarrow \rho(E;S), \qquad \text{as } n \to \infty,$$

provided that E is a continuity point of $\rho(\,\cdot\,;S)$.

We say that $\rho(E;S)$ exists if both $\rho_L^{\mathrm{D}}(E;S)$ and $\rho_L^{\mathrm{N}}(E;S)$ converge and the limits coincide (cf. Eqns. (2.9) and (2.11)). Clearly, this is the case for both periodic and ergodic operators, and so the above Lemma can be applied in our situation.

Proof. First assume that S_n is increasing. Take some $\varepsilon > 0$. By assumption, $\rho(\,\cdot\,;S)$ is continuous in E, hence there is $\gamma > 0$ with

$$\left|\rho(\tilde{E};S) - \rho(E;S)\right| < \varepsilon, \qquad \tilde{E} \in (E, E+2\gamma). \tag{5.9}$$

For some $L_0 > 0$ we have

$$\left|\rho_{L_0}^{\mathrm{N}}(E+\gamma;S) - \rho(E+\gamma;S)\right| < \varepsilon. \tag{5.10}$$

Now consider the operators S_L^{N}, which is S restricted to cubes Q_L with Neumann boundary conditions, as in Chapter 2.3. If E is an eigenvalue of $S_{L_0}^{\mathrm{N}}$, take some $E_0 \in (E, E+\gamma)$ such that E_0 is not an eigenvalue of $S_{L_0}^{\mathrm{N}}$. Note that the spectrum of $S_{L_0}^{\mathrm{N}}$ is countable. Then Eqns. (5.9)–(5.10) together with monotonicity of $\rho_{L_0}^{\mathrm{N}}(\,\cdot\,;S)$ give

$$\left|\rho_{L_0}^{\mathrm{N}}(E_0;S) - \rho(E;S)\right| < 2\varepsilon. \tag{5.11}$$

Since S_n converges to S in strong resolvent sense, compactness (cf. [Kat, Thm. VIII-3.5]) implies $S_{n,L_0}^{\mathrm{N}} \to S_{L_0}^{\mathrm{N}}$ even in norm resolvent sense. As norm resolvent convergence implies convergence of the spectral projections P in norm (cf. [RS1, Thm. VIII.23])

$$\left\|\mathsf{P}_{(-\infty,E_0)}(S_{n,L_0}^{\mathrm{N}}) - \mathsf{P}_{(-\infty,E_0)}(S_{L_0}^{\mathrm{N}})\right\| \longrightarrow 0, \qquad n \to \infty,$$

there is $n_0 \in \mathbb{N}$ with

$$\rho_{L_0}^{\mathrm{N}}(E_0;S_n) - \rho_{L_0}^{\mathrm{N}}(E_0;S) = 0, \qquad n \geq n_0, \tag{5.12}$$

where we took

$$\rho^{\mathrm{N}}_{L_0}(E_0;S) \le \rho^{\mathrm{N}}_{L_0}(E_0;S_{k+1}) \le \rho^{\mathrm{N}}_{L_0}(E_0;S_k), \qquad k \in \mathbb{N},$$

into account. Now, combining Eqns. (5.11)–(5.12) with the monotonicity of $\rho(E,S_n) \le \rho(E_0,S_n)$ and $\rho(E_0;S_n) \le \rho^{\mathrm{N}}_{L_0}(E_0;S_n)$ gives

$$\big|\rho(E;S_n) - \rho(E;S)\big| < 2\varepsilon, \qquad n \ge n_0,$$

which proves the claim.

As for decreasing operators S_n, the same arguments apply by taking Dirichlet boundary conditions S^{D}_L instead of Neumann boundary conditions S^{N}_L. □

Applying the above Lemma to our random operators establishes the almost sure convergence of the density of states.

Proposition 5.8 *Let $T_{\omega,\lambda}$ be as in one of the models M1–M3. Moreover, assume that E is a continuity point of $\rho(\,\cdot\,;-\Delta_{M(\omega)})$ for all $\omega \in \Omega_0 \subset \Omega$, with $P(\Omega_0) = 1$. Then*

$$\rho_\lambda(E) \longrightarrow \rho_\infty(E) = \mathbb{E}\big[\rho(E;-\Delta_{M(\omega)})\big], \qquad \text{as } \lambda \to \infty.$$

Here ρ_λ denotes the almost sure integrated density of states of $T_{\omega,\lambda}$, for $1 \le \lambda < \infty$.

Proof. To apply Lemma 5.7 we note that $T_{\omega,\lambda}$ increases monotonically, and $T_{\omega,\lambda} \to -\Delta_{M(\omega)}$ in strong resolvent sense, as $\lambda \to \infty$. Thus it is sufficient to consider sequences $\lambda_n \to \infty$, as $\mathbb{N} \ni n \to \infty$. For each $n \in \mathbb{N}$, Theorem 2.5 provides a set $\Omega_n \subset \Omega$ of full measure such that

$$\rho(E;T_{\omega_0,\lambda_n}) = \rho_{\lambda_n}(E) = \mathbb{E}\big[\rho(E;T_{\omega,\lambda_n})\big], \qquad \omega_0 \in \Omega_n,\ n \in \mathbb{N}.$$

Taking the countable intersection

$$\tilde{\Omega} = \Omega_0 \cap \bigcap_{n\in\mathbb{N}} \Omega_n$$

we obtain a set $\tilde{\Omega} \subset \Omega$ of probability one, such that $\rho(\,\cdot\,;-\Delta_{M(\omega)})$ is continuous at E and $\rho(E;T_{\omega,\lambda_n}) = \rho_{\lambda_n}(E)$, for $n \in \mathbb{N}$, holds for all $\omega \in \tilde{\Omega}$.

Now, by Lemma 5.7, we see that

$$\rho(E;-\Delta_{M(\omega)}) = \lim_{n\to\infty} \rho(E;T_{\omega,\lambda_n}) = \lim_{n\to\infty} \rho_{\lambda_n}(E),$$

for $\omega \in \tilde{\Omega}$, so that one deduces

$$\mathbb{E}\big[\rho(E;-\Delta_{M(\omega)})\big] = \lim_{n\to\infty} \rho_{\lambda_n}(E),$$

using Lebegue's convergence theorem, since $0 \le \rho_{\lambda_n}(E) \le \rho(E;-\Delta) = CE^{m/2}$ is bounded. □

While for any finite $\lambda < \infty$ Theorem 2.5 provides that the density of states is almost surely constant, this cannot be obtained from Theorem 2.5 if $\lambda = \infty$ because the limit operators $-\Delta_{M(\omega)}$ possibly act in different Hilbert spaces, as ω runs through Ω. Thus, in the above proof we had to carefully avoid applying Theorem 2.5 directly in the limit case.

Now we are well prepared to make our intuition precise that the density of states should concentrate around the Dirichlet eigenvalues of the grains, for λ large.

Theorem 5.9 *The density of states measure μ_λ of models M1–M3 converges, as $\lambda \to \infty$. More precisely, the following holds:*

(a) In model M1 let $a, b \notin \sigma(-\Delta_\Xi)$ and $I = (a,b)$. Then

$$\mu_\lambda(I) \longrightarrow \kappa \cdot \sum_{\delta_k \in I} 1, \qquad \text{as } \lambda \to \infty. \tag{5.13}$$

Here δ_k denotes, as above, the k-th eigenvalue of the Dirichlet Laplacian $-\Delta_\Xi$ on the (non-random) primary grain Ξ, and κ is the intensity of the underlying point process.

(b) Consider model M2, and take $I = (a,b)$ such that a and b do not belong to the spectrum of the (random) primary grain almost surely, i.e.

$$\Gamma\Big(a, b \notin \sigma(-\Delta_{\Xi(\omega)})\Big) = 1.$$

Then we have

$$\mu_\lambda(I) \longrightarrow \mathbb{E}_\Gamma\Big[\sum_{\delta_{k,\omega} \in I} 1\Big] = \sum_k \Gamma(\delta_{k,\omega} \in I), \qquad \text{as } \lambda \to \infty, \tag{5.14}$$

where $\delta_{k,\omega}$ denotes the k-th eigenvalue of the (random) Dirichlet Laplacian $-\Delta_{\Xi(\omega)}$ on $\Xi(\omega)$.

(c) Consider model M3. Again, let $I = (a,b)$ with

$$(\Lambda \otimes \Gamma)\Big(a, b \notin \sigma(-\Delta_{S_{r(\omega)}\Xi(\omega)})\Big) = 1,$$

Λ the mark distribution of the typical radius r, and Γ the distribution of the primary grain Ξ, and S the scaling operator (see Eqn. (3.11)). Then

$$\mu_\lambda(I) \longrightarrow \kappa \cdot \sum_k \mathbb{E}_\Gamma\big[\, \Lambda(U_{k,\omega}) \,\big] \qquad \text{as } \lambda \to \infty, \tag{5.15}$$

with

$$U_{k,\omega} = \Big(\big(\delta_{k,\omega}/b\big)^{1/2}, \big(\delta_{k,\omega}/a\big)^{1/2} \Big) \subset \mathbb{R},$$

and notation as in part (b).

Proof. In view of Proposition 5.8 and $T_{\omega,\infty} = -\Delta_{M(\omega)}$ it is clearly enough to determine the value of $\mu_\infty(I) = \rho_\infty(b) - \rho_\infty(a)$ where $\rho_\infty(E) = \mathbb{E}\big[\rho(E; -\Delta_{M(\omega)})\big]$. Note that in all three cases the assumptions are chosen to meet the requirements of Proposition 5.8.

(a) In case of model M1, for any $\omega \in \Omega$, the limit operator $-\Delta_{M(\omega)}$ is given by the direct sum $\bigoplus_{n\in\mathbb{N}} -\Delta_{M_n(\omega)}$. But, as the spectrum of $-\Delta_{M_n(\omega)}$ coincides with $\sigma(-\Delta_\Xi) = \{\delta_k \mid k \in \mathbb{N}\}$, independently of both n and ω, we get

$$\dim \mathsf{P}_{(-\infty,E)}(-\Delta_{M_n(\omega)}) = \dim \mathsf{P}_{(-\infty,E)}(-\Delta_\Xi) = \sum_{\delta_k < E} 1, \qquad n \in \mathbb{N},\ \omega \in \Omega.$$

As usual, $\dim \mathsf{P}_{(-\infty,E)}(S)$ denotes the dimension of the spectral projection $\mathsf{P}_{(-\infty,E)}$ of an operator S which equals the number of eigenvalues of S below E. Furthermore, $\rho(E; -\Delta_{M(\omega)})$ is almost surely constant and the expected number of grains per volume element equals κ, and so

$$\mathbb{E}\big[\rho(E; -\Delta_{M(\omega)})\big] = \kappa \cdot \mathbb{E}\big[\dim \mathsf{P}_{(-\infty,E)}(-\Delta_\Xi)\big] = \kappa \cdot \sum_{\delta_k < E} 1,$$

which proves the claim.

(b) The grains $\Xi_i(\omega)$ are i.i.d., thus

$$\mathbb{E}_\Gamma\big[\dim \mathsf{P}_{(-\infty,E)}(-\Delta_{M_i(\omega)})\big] = \mathbb{E}_\Gamma\big[\dim \mathsf{P}_{(-\infty,E)}(-\Delta_{\Xi(\omega)})\big], \quad i \in \mathbb{Z}^m,$$

implies

$$\mathbb{E}\big[\rho(E; -\Delta_{M(\omega)})\big] = \mathbb{E}_\Gamma\Big[\sum_{\delta_{k,\omega} < E} 1\Big],$$

proving (b).

(c) For the soft-core process $\Phi(\omega) = \{x_n(\omega)\}_{n\in\mathbb{N}}$ denote by $A_L(\omega)$ the number of points $x_n(\omega)$ within Q_L. Since $\Phi(\omega)$ is ergodic we get (cf. [SKM])

$$\frac{A_L(\omega)}{L^m} \longrightarrow \kappa \quad P\text{-a.s.}, \qquad \text{as } L \to \infty. \tag{5.16}$$

Without restriction, we may assume that $x_n(\omega) \in Q_L$ if and only if $1 \le n \le A_L(\omega)$. Let $\delta_{k,\omega}^{(n)}$ denote the k-th eigenvalue of the Dirichlet Laplacian $-\Delta_{M_n(\omega)}$ where $M_n(\omega) = S_{r_n(\omega)}\Xi_n(\omega)$. The strong law of large numbers holds for ergodic random fields $Z_n(\omega) = \dim \mathsf{P}_{(-\infty,E)}(-\Delta_{M_n(\omega)}) = \sum_{\delta_{k,\omega}^{(n)}<E} 1$ (see, e.g., [Ki2]), and so

$$\frac{1}{A_L(\omega)} \sum_{n=1}^{A_L(\omega)} Z_n(\omega) \longrightarrow \mathbb{E}_{\Lambda\otimes\Gamma}\big[Z_1\big] \quad P\text{-a.s.}, \quad \text{as } L \to \infty. \tag{5.17}$$

Thus combining Eqns. (5.16) and (5.17) gives, for $\omega \in \Omega$,

$$\begin{aligned}
\rho(E; -\Delta_{M(\omega)}) &= \lim_{L\to\infty} \rho_L^{\mathsf{D}}(E; -\Delta_{M(\omega)}) \\
&= \lim_{L\to\infty} \frac{1}{L^m} \dim \mathsf{P}_{(-\infty,E)} \left(\bigoplus_{n=1}^{A_L(\omega)} -\Delta_{M_n(\omega)} \right) \\
&= \lim_{L\to\infty} \frac{A_L(\omega)}{L^m} \cdot \frac{1}{A_L(\omega)} \sum_{n=1}^{A_L(\omega)} \left(\sum_{\delta_{k,\omega}^{(n)}<E} 1 \right) \\
&= \kappa \cdot \mathbb{E}_{\Lambda\otimes\Gamma}\left[\sum_{\delta_{k,\omega}<E} 1 \right] \qquad P\text{-a.s.},
\end{aligned}$$

where $\delta_{k,\omega} = \delta_{k,\omega}^{(1)}$. Therefore we see that $\rho(E; -\Delta_{M(\omega)})$ is P-a.s. constant, and we obtain

$$\mathbb{E}_P\left[\rho(E; -\Delta_{M(\omega)})\right] = \kappa \cdot \mathbb{E}_{\Lambda\otimes\Gamma}\left[\sum_{\delta_{k,\omega}<E} 1 \right].$$

As S_r is just scaling, some E is an eigenvalue of $-\Delta_\Xi$ if and only if $r^{-2}E$ is an eigenvalue of $-\Delta_{S_r\Xi}$. So we can write

$$\mathbb{E}_P\left[\rho(E; -\Delta_{M(\omega)})\right] = \kappa \cdot \mathbb{E}_{\Lambda\otimes\Gamma}\left[\sum_{r^{-2}(\omega_1)\delta_{k,\omega_2}<E} 1 \right],$$

where we denote $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$ with Λ the probability measure in Ω_1 and Γ the probability measure in Ω_2. Now, using Fubini, we obtain

$$\begin{aligned}
\mathbb{E}_P\left[\rho(E; -\Delta_{M(\omega)})\right] &= \\
&= \kappa \cdot \sum_k \int \chi_{\{r^{-2}(\omega_1)\delta_{k,\omega_2} < E\}}\, d(\Lambda \otimes \Gamma) \\
&= \kappa \cdot \sum_k \int \left(\int \chi_{\{r(\omega_1) > (\delta_{k,\omega_2}/E)^{1/2}\}}\, d\Lambda(\omega_1) \right) d\Gamma(\omega_2) \\
&= \kappa \cdot \sum_k \mathbb{E}_\Gamma \left[\Lambda\left(r > \left(\delta_{k,\omega_2}/E\right)^{1/2} \right) \right],
\end{aligned}$$

which proves (c). □

For a deeper look at the concentration regions of the density of states of model M3 let

$$\delta_k^{\min} = \min_{\omega \in \operatorname{supp}\Lambda} r^{-2}(\omega) \cdot \inf_{\omega \in \operatorname{supp}\Gamma} \delta_{k,\omega}$$

and

$$\delta_k^{\max} = \max_{\omega \in \operatorname{supp}\Lambda} r^{-2}(\omega) \cdot \sup_{\omega \in \operatorname{supp}\Gamma} \delta_{k,\omega},$$

for $k \in \mathbb{N}$. With this notation $\bigcup_{k \in \mathbb{N}} \left[\delta_k^{\min}, \delta_k^{\max}\right]$ encloses the parts of the real line in which μ_λ concentrates its mass, as $\lambda \to \infty$. Figure 5.3 shows an attempt to illustrate the density $\frac{d}{dE}\rho_\lambda(E)$ of the integrated density of states $\rho_\lambda(E)$. The shaded area suggests the spectrum $\Sigma_\lambda = [0, \infty)$.

Part (b) of Theorem 5.9 has a special form in case of model M2'. Let δ_k denote the Dirichlet eigenvalues of $-\Delta_\Xi$, the Dirichlet Laplacian on the unscaled star-shaped primary grain Ξ. Then the same argument as in the proof of Theorem 5.9(c) implies

$$\mu_\lambda\big((a,b)\big) \longrightarrow \sum_k \Gamma\left(\left((\delta_k/b)^{1/2}, (\delta_k/a)^{1/2} \right) \right), \qquad \text{as } \lambda \to \infty.$$

Remark 5.10 (a) While Theorem 2.5 establishes the almost sure non-randomness of the densities of states only for $\lambda < \infty$, we obtain from Theorem 5.9 that this holds also for $\lambda = \infty$.

(b) The formulas in Eqns. (5.13)–(5.15) for the densities of states in the limit case suggest a connection with the concept of Palm distributions (see [SKM]).

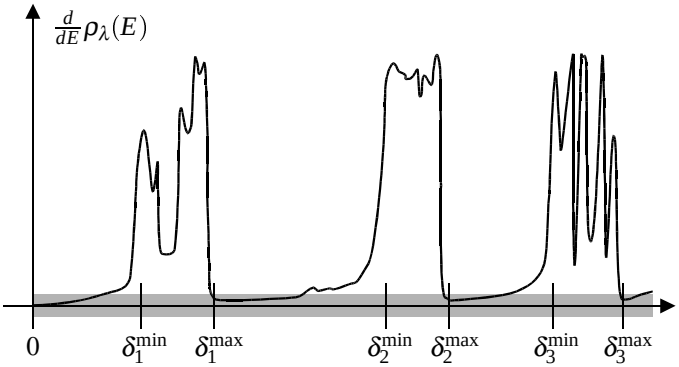


Figure 5.3: Density function of the integrated density of states ρ_λ of model M3 for large λ.

This means that the behaviour in the limit case can be expressed completely in terms of properties of a typical grain together with the intensity κ of the point process. Here *typical grain* stands for the (non-random) primary grain Ξ (model M1) or the (random) primary grain $\Xi(\omega)$ and $S_{r(\omega)}\Xi(\omega)$ (models M2 and M3), respectively. The (random) positions of the grains do not contribute to the limit densities at all.

(c) We have seen in Theorem 5.1 that neither model M1 nor model M3 provides spectral gaps. But Theorem 5.9 suggests that we have to look more carefully at the definition of gaps. In experimental science one can detect spectrum in a given (energy) interval I only if it has a sufficiently high density there. If this is not the case such intervals are called *pseudo-gaps*. Thus, for λ large, Theorem 5.9 establishes the existence of pseudo-gaps in models M1 and M3 in intervals I that do not contain any Dirichlet eigenvalue δ_k (model M1) and $\delta_{k,\omega}$ (model M3), respectively. By the same argument, we see that in model M2' the pseudo-gaps are larger than the real gaps given by Corollary 5.5.

Summarizing, we have proven in this Chapter that only in model M2 there is a chance to have gaps in the spectrum, while models M1 and M3 do not exhibit any gaps at all. Moreover, in case of model M2 the gaps can be expressed in a similar way as in the periodic case. On the other hand, the density of states measure concentrates around the eigenvalues of the Dirichlet Laplacian $-\Delta_\Xi$ (model M1), $-\Delta_{\Xi(\omega)}$ (model M2), and $-\Delta_{S_{r(\omega)}\Xi(\omega)}$ (model M3), for λ large. Especially in case of model M1 it is remarkable that these concentra-

tion points are non-random and coincide with the situation of the periodic case in Chapter 4. For models M2 and M3, due to averaging, these concentration points (or regions) will be in general non-degenerate intervals.

A Appendix

A.1 Measurability of operators

In this appendix we examine the measurability of $T_{\omega,\lambda}$, the operators of models M1–M3, which is necessary for $T_{\omega,\lambda}$ to be random operators (cf. Definition 2.2). We will need the following Lemma from [FKl1].

Lemma A.1 *Let T_ω be a family of almost surely s.a. operators in a separable Hilbert space $\mathcal{H}$, such that*

$$S \leq T_\omega \leq cS \quad \textit{almost surely}$$

for some non-negative s.a. operator S and some constant $c < \infty$. Furthermore denote by $\mathbf{t}_\omega$ and $\mathbf{s}$ the corresponding quadratic forms, and let $\mathcal{Q} \subset \mathcal{H}$ be the common form domain, almost surely.

Suppose $\omega \mapsto \mathbf{t}_\omega[u]$ is measurable for all $u \in \mathcal{Q}$, then T_ω is measurable, i.e. T_ω is a random operator.

Proof. See Theorem 38 in [FKl1]. □

Theorem A.2 *Each family of operators $T_{\omega,\lambda}$ from models M1–M3 is a random operator.*

Proof. Because of $-\Delta \leq T_{\omega,\lambda} \leq -\lambda\Delta$ we can apply the above Lemma with $\mathcal{Q} = \mathcal{H}^1(\mathbb{R}^m)$. It remains to show that, for any $u \in \mathcal{H}^1(\mathbb{R}^m)$, the mapping $\omega \mapsto \mathbf{t}_{\omega,\lambda}[u]$ is measurable. But this is true since $\omega \mapsto M(\omega)$ is measurable, and, therefore, $\omega \mapsto a_{\omega,\lambda}$, too. □

A.2 Convergence of operators

Here we give the promised proof of Lemma 3.3 which relies on the following result.

Lemma A.3 *Consider selfadjoint operators S and S_n, $n \in \mathbb{N}$, in a Hilbert space $\mathcal{H}$. Moreover, assume that there are two sequences of selfadjoint operators $S_n^\pm$ in $\mathcal{H}$ and positive constants $\alpha^- < 1$ and $\alpha^+ < \infty$ such that*

$$S - S_n^- \leq S_n \leq S + S_n^+ \tag{A.1}$$

and

$$0 \le S_n^{\pm} \le \alpha^{\pm} S, \tag{A.2}$$

for all $n \in \mathbb{N}$*. If, for a form core* $\mathcal{D}$ *of* S*,*

$$\lim_{n\to\infty} \mathbf{s}_n^{\pm}[u] = 0, \qquad \textit{all } u \in \mathcal{D}, \tag{A.3}$$

($\mathbf{s}_n^{\pm}$ *the corresponding quadratic forms) then*

$$S_n \longrightarrow S \quad \textit{in s.r.s.},$$

as $n \to \infty$*.*

For a proof see [FKl1], Lemma 45. The main improvement we obtain from Lemma A.3 is that we only need to deal with quadratic forms instead of operators when we want to check convergence in strong resolvent sense. With this preparation we can prove Lemma 3.3 which we restate for convenience.

Lemma 3.3 *For* $T_{\omega,\lambda}$ *as in models M1–M3 suppose* $\omega_n \to \omega_0$ *in* $L_{1,\mathrm{loc}}(\mathbb{R}^m)$*. Then*

$$T_{\omega_n,\lambda} \longrightarrow T_{\omega_0,\lambda} \quad \textit{in s.r.s.},$$

and in particular

$$\sigma(T_{\omega_0,\lambda}) \subset \overline{\bigcup_{n\in\mathbb{N}} \sigma(T_{\omega_n,\lambda})} \tag{A.4}$$

holds.

Proof. Fix some $\lambda \ge 1$, and suppose $\omega_n \to \omega_0$ in $L_{1,\mathrm{loc}}(\mathbb{R}^m)$. Let

$$b_n^+(x) = \max\left\{0,\, a_{\omega_n,\lambda}(x) - a_{\omega_0,\lambda}(x)\right\} \ge 0$$
$$b_n^-(x) = -\min\left\{0,\, a_{\omega_n,\lambda}(x) - a_{\omega_0,\lambda}(x)\right\} \ge 0,$$

and so

$$a_{\omega_0,\lambda} - b_n^- \le a_{\omega_n,\lambda} \le a_{\omega_0,\lambda} + b_n^+$$

implies

$$T_{\omega_0,\lambda} - S_n^- \le T_{\omega_n,\lambda} \le T_{\omega_0,\lambda} + S_n^+$$

where $S_n^{\pm} = \nabla^* b_n^{\pm} \nabla$, and sums of operators being understood as sums of the associated forms. For each $\varphi \in \mathbb{C}_c^{\infty}(\mathbb{R}^m)$ there is some cube Q_φ such that $\operatorname{supp}\varphi \subset Q_\varphi$, thus

$$0 \le \mathbf{s}_n^{\pm}[\varphi] = \int_{Q_\varphi} b_n^{\pm} |\nabla\varphi|^2 \le C_\varphi \left\| b_n^{\pm} \right\|_{L_1(Q_\varphi)} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Up to now, conditions (A.1) and (A.3) of Lemma A.3 are fulfilled. As for condition (A.2), choosing $\alpha^+ = \lambda < \infty$ and $\alpha^- = 1 - \frac{1}{\lambda} < 1$ one easily verifies condition (A.2), e.g.

$$\begin{aligned} b_n^- &= -\min\{0, a_{\omega_n,\lambda} - a_{\omega_0,\lambda}\} \;=\; a_{\omega_0,\lambda} - \min\{a_{\omega_0,\lambda}, a_{\omega_n,\lambda}\} \\ &\le a_{\omega_0,\lambda} - 1 \;\le\; (1 - \tfrac{1}{\lambda})\, a_{\omega_0,\lambda}. \end{aligned}$$

As $C_c^\infty(\mathbb{R}^m)$ is a form core of $T_{\omega_0,\lambda}$ Lemma A.3 implies $T_{\omega_n,\lambda} \to T_{\omega_0,\lambda}$ in strong resolvent sense, and by standard arguments we conclude Eqn. (A.4) (e.g. see [RS1]). □

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Zusammenfassung

Die vorliegende Arbeit untersucht Spektraleigenschaften einer Klasse von selbstadjungierten Operatoren vom Divergenztyp mit zufälligen Koeffizienten

$$T_\omega = \nabla^* a_\omega(x)\nabla \quad \text{in } L_2(\mathbb{R}^m)$$

für Raumdimensionen $m \geq 2$. Die zufälligen Koeffizienten werden hierbei rein geometrisch modelliert: auf einer zufälligen Menge $M(\omega) \subset \mathbb{R}^m$ nehmen die skalarwertigen Koeffizienten den festen Wert 1, sonst den variablen Wert λ an, wobei $\lambda \gg 1$ gewählt wird. Da $\mathbb{R}^m \setminus M(\omega)$ als zusammenhängend vorausgesetzt wird, kann $M(\omega)$ als Störung eines Grundmediums interpretiert werden. In Bezug auf Wärmeleitung könnte man $M(\omega)$ als gering leitende Sandkörner oder Luftblasen in einem stark leitenden Metall ansehen. Weitere Anwendungen finden sich in der Akustik und bei optischen Kristallen (optische Halbleiter). Die Problemstellung ist wesentlich durch eine Arbeit von R. Hempel und K. Lienau (1998) motiviert, in der eine vergleichbare Situation für periodische Medien behandelt wird.

Zunächst werden geeignete stochastische Modelle für die Beschreibung der aus einzelnen Körnern (engl. grains) bestehenden Störmenge $M(\omega)$ diskutiert. Da alle vorgestellten Modelle zu ergodischen Operatoren führen, sind jeweils sowohl das Spektrum als auch die Zustandsdichte fast sicher unabhängig von ω und somit nicht zufällig. Das Verhalten dieser beiden spektraltheoretischen Größen wird untersucht, insbesondere die Massekonzentration des Zustandsdichtemaßes für große Parameter $\lambda \gg 1$ und den Limesfall $\lambda \to \infty$. Das Konzentrationsverhalten wird bestimmt durch stochastische Mittelbildung von Dirichlet-Problemen auf den das Grundmedium verunreinigenden Körnern. Es wird gezeigt, dass sich das Verhalten des Spektrums und der Zustandsdichte im Limes $\lambda \to \infty$ durch Eigenschaften *typischer* Körner ausdrücken lässt, und sich somit in das Konzept der Palm-Verteilungen einreiht.

Damit verbindet die vorliegende Arbeit zwei Bereiche der Spektraltheorie, die in der Literatur bereits mehrfach untersucht worden sind. Zum einen sind dies Grenzeigenschaften von Operatoren im starken Kopplungslimes (d.h. $\lambda \to \infty$) und zum anderen Spektraleigenschaften zufälliger Operatoren.

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